Intrinsicity, Rigidity, Reconstructability

ver. 1.1, astérisque[1]

May 1, 2022

[1] Twitter@luculentissima

Goals of This Talk

In this talk, we look at various **basic topics**, keeping in mind the background of **anabelian**, in particular **mono-anabelian geometry**. Even seemingly unrelated topics are all connected by the background of anabelian geometry.

It is difficult to formulate rigorously what is meant by

reconstructable in a mono-anabelian fashion

or

reconstructable in a purely group theoretic fashion

et c. (cf. **species, mutations**, [IUTeichIV]), but by looking through various concrete examples, it would be clear that such reconstructions do exist!

Since there is little point in going over the proofs of a relatively niche topic in detail, I will focus on

showing the key points of the strategy

with regard to the proofs.

Research Background

As an undergraduate student, I once lost all faith in mathematical symbol manipulation.

What is difference?

What differences are reasonable to accept or recognize?

I asked myself.

I wanted interpretations to be **intrinsically derived** from **the real**. I did not want interpretations to depend on how I perceive them. I wanted to interpret it without reference to anything but the most certain and pure "itself".

Table of Contents

Is Set Theoretic Reconstruction Enough?

Assume that we are given a polynomial ring $R[X_1, \cdots, X_n]$ with coefficient ring R. Can we then reconstruct **purely ring-theoretically** the objects R and n for example?

Can we say that R is the coefficient ring of $R[X_1, \cdots, X_n]$ *simply because we see the symbol* "R", or that the number of indeterminates is n *simply because we see the symbol* "n"?

$\Longrightarrow$For example, if we define the notion of a **map** as a triple $(X, Y, \Gamma \subseteq X \times Y)$, its **domain** and **codomain** can always be reconstructed from the map itself. Then, if we define $R[X_1, \cdots, X_n]$ to be a *-tuple of a set of maps $\mathbb{N}^n \to R$ and such-and-such, both R and n can always be reconstructed from the set $R[X_1, \cdots, X_n]$.
$\Longrightarrow$Is this enough? Maybe NO.

Reference Models

For a general ring S, let us consider a suitable definition of the abstract notion "S is a polynomial ring" : a ring S is a **polynomial ring** if there exists a ring R, $n \in \mathbb{N}$ and a ring isomorphism(i.e., **reference isomorphism**) such that $S \simeq R[X_1, \cdots, X_n]$.

$\Longrightarrow$Here, are R and n unique?
$\Longrightarrow$NO. For example, let S to be the zero ring. Then, we have infinitely many possible choices for n. That is to say, the choice of a "model" $R[X_1, \cdots, X_n]$ of S is not intrinsic.

$\Longrightarrow$Once we fix one model, we can reconstruct R and n from the set-theoretic object $R[X_1, \cdots, X_n]$ itself by the previous argument, but we want to reconstruct R and n **solely from S in a purely ring-theoretic way**, without using, and independent of, such a particular model choice.

We shall say a ring R is **of FC type** if there exists a model polynomial ring for R over some field, i.e., there exists k, $n \in \mathbb{N}$ and an isomorphism such that $k[X_1, \cdots, X_n] \simeq R$

Proposition

Let R be a ring of FC type. Then, we may reconstruct "k" and "n" solely from R in a purely ring-theoretical manner.

Proof.

For given ring of FC type R, let $k^\times := R^\times$, $k := \{0\} \cup k^\times$ and regard k as a ring by using the induced ring structure from R. Also let $n := \dim(R)$ then, there exists a natural ring isomorphism $R \simeq k[X_1, \cdots, X_n]$, up to S_n, and we win. □

In what follows, we will examine this proof a little more closely.

Krull Dimension

Definition (Krull dimension)

Let R be a ring. Then, we define $\dim(R)$ to be the supremum of the set of natural numbers $n \in \mathbb{N}$ such that there exists an ascending chain of length n in R.

Remark

The notion of Krull dimension is **ring-theoretic**. In particular, the existence of a ring isomorphism $R \simeq S$ implies $\dim(R) = \dim(S)$.

Lemma

Let k be a field. Then, $\dim(k[X_1, \cdots, X_n]) = n$*.*

First, observe that from the existence of an ascending chain,

$$0 \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \cdots \subsetneq (X_1, \cdots, X_n)$$

the inequality $n \leq \dim(k[X_1, \cdots, X_n])$ is shown easily. The converse inequality is non-trivial.

How to Calculate Krull Dimensions

There are various proof strategies. For example, using facts listed below, we can inductively show that $\dim(k[X_1, \cdots, X_n]) = n$:

1. For a Noetherian ring R, $R[X]$is also Noetherian and $\dim(R[X]) = \dim(R) + 1$ holds.
2. A field k is Noetherian and $\dim(k) = 0$.

How to Calculate Krull Dimensions

We also have a more direct way to show the equality. Letting $R := k[X_1, \cdots, X_n], \overline{R} := \overline{k}[X_1, \cdots, X_n]$, we can show properties listed below:

1. The natural map $\mathrm{Specm}(\overline{R}) \to \mathrm{Specm}(R)$ is bijective.
2. For any $\overline{\mathfrak{m}} \in \mathrm{Specm}(\overline{R})$, let $\mathfrak{m}$ be the image of $\overline{\mathfrak{m}}$ then, $\dim(R_{\mathfrak{m}}) = \dim(\overline{R}_{\overline{\mathfrak{m}}})$.
3. For $\overline{\mathfrak{m}} := (X_1, \cdots, X_n) \in \mathrm{Specm}(\overline{R})$, $\dim(\overline{R}_{\overline{\mathfrak{m}}}) = n$.
4. For any $\overline{\mathfrak{m}}, \overline{\mathfrak{n}} \in \mathrm{Specm}(\overline{R})$, there exists a natural isomorphism $\overline{R}_{\overline{\mathfrak{m}}} \simeq \overline{R}_{\overline{\mathfrak{n}}}$. In particular, $\dim(\overline{R}_{\overline{\mathfrak{m}}}) = \dim(\overline{R}_{\overline{\mathfrak{n}}})$.

1, 2 follow from well-known arguments around the theory of **integral extensions, going-up, going-down**. 4 is essentially **Hilbert's Nullstellensatz**. 3 follows from well-known arguments around the proof of dimension theorem.

Image of an Algorithm

We would like to consider **reconstruction algorithms** themselves as mathematical objects (cf. species, mutations), but for the time being we will consider them in terms of categories, i.e., "**categorified**" objects.

Definition

Let $\mathbf{R}$ the full subcategory of $\mathbf{Rng}$ whose objects are rings R such that $\dim(R) < \infty$ and the ring structure on R can "descend" to $R^\times \cup \{0\} \subseteq R$.

Image of an Algorithm

In this case, the algorithm we have just seen determines a functor $\mathbf{R} \to \mathbf{Fld} \times \mathbb{N}$ and together with the functor induced by the algorithm "to construct a polynomial ring",

$$\Phi : \mathbf{R} \to \mathbf{Fld} \times \mathbb{N} \to \mathbf{R}.$$

Now it is obvious that $\Phi \circ \Phi \simeq \Phi$. As $\mathbf{Fld} \times \mathbb{N} \to \mathbf{R} \to \mathbf{Fld} \times \mathbb{N}$ is isomorphic to the identity functor, by the standard argument of **split idempotent morphisms**, one can reconstruct $\mathbf{Fld} \times \mathbb{N}$ from Φ up to category equivalence in a purely category theoretic manner.

Question

What is the relation between R and $\Phi(R)$?

The interpretation introduced above may be applied to the theory of primary ideal decomposition.

Fix a **Dedekind ring** R. First, define a **primary ideal decomposition** of I to be a finite set of primary ideals $\{\mathfrak{q}_\lambda\}$ of R such that $\cap_\lambda \mathfrak{q}_\lambda = I$. Next, we shall say I is **decomposable** if I has a primary ideal decomposition.

At this point, we observe the following analogies:

abstract input object	reconstructable objects
of FC type polynomial ring R	k, n
decomposable ideal I	$\{q_\lambda\}$

In the latter case, the so-called uniqueness of the prime factorization follows.

Primary Ideal Decomposition

Firstly, let us review the theory of primary ideals. It is immediate to see the followings are equivalent:

Lemma

Let R be a ring, $\mathfrak{q} \subsetneq R$ be a proper ideal of R. Then, the followings are equivalent:

1. $\forall x, y \in R, xy \in \mathfrak{q} \implies x \in \mathfrak{q} \vee \exists n > 0, y^n \in \mathfrak{q}$.
2. Any zero divisor in $R/\mathfrak{q}$ is nilpotent.

Any ideal $\mathfrak{q}$ which satisfies the equivalent conditions above is called **primary**. A primary ideal $\mathfrak{q}$ which satisfies $\sqrt{\mathfrak{q}} = \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$, then $\mathfrak{q}$ is also called $\mathfrak{p}$-**primary**.

Primary Ideal Decomposition

A primary ideal decomposition $\{\mathfrak{q}_\lambda\}$ of an ideal I of R is called **minimal** if the followings hold:

1. $\sqrt{\mathfrak{q}_\lambda}$ are mutually distinct.
2. For all λ, $\cap_{\lambda' \neq \lambda} \mathfrak{q}_{\lambda'} \not\subseteq \mathfrak{q}_\lambda$.

For arbitrary ideal I, the existence of decomposition of I is equivalent to the existence of minimal decomposition of I. This follows from the fact that $\mathfrak{p}$-primary ideals are **stable under intersection**.

For an ideal I, let $\mathrm{Assoc}(I) := \{\sqrt{(I : x)} : \sqrt{(I : x)} \in \mathrm{Spec}(R), x \in R\}$. It is immediate to observe that the notion $(I : x)$ and $\sqrt{I}$ are **compatible with intersection** and this immediately implies that if $\{\mathfrak{q}_\lambda\}_{\lambda \in \Lambda}$ is a minimal decomposition of I,

$$\mathrm{Assoc}(I) = \{\sqrt{\mathfrak{q}_\lambda}\}_{\lambda \in \Lambda},$$

i.e., "$\mathrm{rad}(I)$" may be **intrinsically defined**.

Reconstruction of Primary Components

A set Σ is (R, I)-**isolated** if the followings are satisfied:

1. $\Sigma \subseteq \mathrm{Assoc}(I)$.
2. $\forall \mathfrak{p}' \in \mathrm{Assoc}(I), \forall \mathfrak{p} \in \Sigma[\mathfrak{p}' \subseteq \mathfrak{p} \implies \mathfrak{p}' \in \Sigma]$.

For any (R, I)-isolated set Σ, one may associate corresponding multiplicatively closed subset $S_\Sigma := R \setminus \cup_{\mathfrak{p} \in \Sigma} \mathfrak{p}$.

Now we can see how to reconstruct minimal decomposition $\{\mathfrak{q}_\lambda\}$ of a decomposable ideal I:

1. Let $\Sigma_\mathfrak{p} := \{\mathfrak{p}\}$ for each $\mathfrak{p} \in \mathrm{Minimal}(\mathrm{Assoc}(I))$.
2. Then, $\Sigma_\mathfrak{p}$ is (R, I)-isolated.
3. Taking the contraction of the extension of I along $R \to S_{\Sigma_\mathfrak{p}}^{-1} R$ for each $\mathfrak{p} \in \mathrm{Minimal}(\mathrm{Assoc}(I))$, one obtains a set which is equal to $\mathrm{Minimal}(\{\mathfrak{q}_\lambda\}) = \{\mathfrak{q}_\lambda\}$.

Problems

Raising some problems here:

The example we have just seen is to reconstruct the reference model up to isomorphism for an object such that there exists a reference isomorphism to the reference model.

$\Longrightarrow$ Can other objects which are constructed from the reference model also be reconstructed? Also, if greater **indeterminacies** are allowed, can more objects be reconstructed? In such a case, what are the conditions for reducing indeterminacies?

We would like to consider the intrinsic conditions that leads to these reconstructability.

Group of MLF Type

Definition

1. We call a finite extension $\mathbb{Q}_p \to K$ an **MLF**. For an MLF K, $G_K := \mathrm{Gal}(\overline{K}/K)$.
2. A group G is **of MLF type** if there exists an MLF K and an isomorphism $G \simeq G_K$.

Lemma

1. In general, topological groups that arises from MLF are topologically finitely generated([NSW]).
2. As open subgroups of a profinite group which is topologically finitely generated are the same as of finite index subgroups([NS]), one may reconstruct a topological group G from an abstract group G.
3. In topologically finitely generated profinite groups, abstract commutator subgroups are closed([NS]), i.e., it is not necessary to take the closure.

Mono-Anabelian Reconstructions via LCFT

First, let us look at the case $K = \mathbb{Q}_p$. Now, observe the following properties:

1. $G_p^{\mathrm{ab}} \simeq \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$.
2. By local class field theory, $G_p^{\mathrm{ab}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times$.
3. $(G_p^{\mathrm{ab}})_{\mathrm{tor}} \simeq (\mathbb{Z}_p^\times)_{\mathrm{tor}}$.
4. $G_p^{\mathrm{ab/tor}} := G_p^{\mathrm{ab}}/(G_p^{\mathrm{ab}})_{\mathrm{tor}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)_{\mathrm{tor}}$.
5. $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)_{\mathrm{tor}} \simeq \mathbb{Z}_p$.
6. $\widehat{\mathbb{Z}} \simeq \prod_{q\in\mathrm{Spec}(\mathbb{Z})} \mathbb{Z}_q$.

Thus,

$$G_p^{\mathrm{ab/tor}} \simeq \prod_{q\in\mathrm{Spec}(\mathbb{Z}),\ q\neq p} \mathbb{Z}_q \times \mathbb{Z}_p^2$$

holds. Therefore, in the case $\ell = p$, $G_p^{\mathrm{ab/tor}}/\ell G^{\mathrm{ab/tor}} \simeq (\mathbb{Z}/\ell\mathbb{Z})^2$, and in the case $\ell \neq p$, $G_p^{\mathrm{ab/tor}}/\ell G^{\mathrm{ab/tor}} \simeq \mathbb{Z}/\ell\mathbb{Z}$.

Mono-Anabelian Reconstructions via LCFT

If $[K : \mathbb{Q}_p] > 1$:

1. $G_K^{\mathsf{ab}} \simeq \mathrm{Gal}(K^{\mathsf{ab}}/K)$.
2. By local class field theory, $G_K^{\mathsf{ab}} \simeq \widehat{\mathbb{Z}} \times \mathcal{O}_K^{\times}$.
3. $\mathcal{O}_K^{\times}/(\mathcal{O}_K^{\times})_{\mathsf{tor}} \simeq \log(1 + \mathfrak{m}_K) \simeq \mathcal{O}_K \simeq \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$.
4. $G^{\mathsf{ab/tor}} := G_K^{\mathsf{ab}}/(G_K^{\mathsf{ab}})_{\mathsf{tor}} \simeq \widehat{\mathbb{Z}} \times \mathbb{Z}_p^{[K:\mathbb{Q}_p]}$.

Therefore, in the case $\ell = p$, $G^{\mathsf{ab/tor}}/\ell G^{\mathsf{ab/tor}} \simeq (\mathbb{Z}/\ell\mathbb{Z})^{[K:\mathbb{Q}_p]+1}$, and in the case $\ell \neq p$, $G^{\mathsf{ab/tor}}/\ell G^{\mathsf{ab/tor}} \simeq \mathbb{Z}/\ell\mathbb{Z}$.

Mono-Anabelian Reconstructions via LCFT

We shall continue to consider reconstruction algorithms of other objects:

1. $d(G) := \log_p \left(\left| G_{\mathsf{ab/tor}} / p G_{\mathsf{ab/tor}} \right| \right) - 1.$
2. $f(G) := \log_p \left(1 + \left| (G^{\mathsf{ab-tor}})^{p'} \right| \right)$. Here $(G^{\mathsf{ab-tor}})^{p'}$ is the p-Sylow subgroup of $G^{\mathsf{ab-tor}}$.
3. $e(G) := d(G)/f(g)$.
4. $I(G) := \bigcap N$. Here, N is an open normal subgroup of G such that $e(G) = e(G)$.
5. $P(G) := \bigcap N$. Here, N is an open normal subgroup of G such that $e(N)/e(G) > 0$ and $e(N)/e(G)$ is prime to p.

Then, once one fixes a reference model, these are isomorphic to the well-known objects, i.e.,
$d_K := [K, \mathbb{Q}_p], f_K := [\mathcal{O}_K/\mathfrak{m}_K, \mathbb{F}_p], e_K := \left| K^\times / \mathcal{O}_K^\times \cdot \mathbb{Q}_p^\times \right|$,
inertia/wild inertia subgroup.

Mono-Anabelian Reconstructions via LCFT

Reconstruction continued:

1. Let $\mathrm{Frob}(G) \in G/I(G)$ to be the unique element such that the conjugate action on $I(G)/P(G)$ is the action determined multiplying by $p^{f(G)}$.
2. $k^\times(G) := G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}}$.
3. $\overline{k}^\times(G) := \lim_{H \subseteq G} k^\times(H)$, Here H is an open subgroup.
4. $\mu(G) := \overline{k}^\times(G)_{\mathrm{tor}}$.
5. $\Lambda(G) := \lim \mu(G)[n]$.

Neukirch-Uchida Theorem

From the previous analogy, one might think that it is possible to recover k from an abstract group G_k. But this is wrong!([JR]) For MLF of residual characteristic $\neq 2$,

$$\mathrm{Isom}(\overline{k}_1/k_1, \overline{k}_2/k_2) \overset{\text{non-isom.}}{\hookrightarrow} \mathrm{Isom}(G_{k_1}, G_{k_2}).$$

In particular,

$$\mathrm{Aut}(\overline{k}/k) \overset{\text{non-isom.}}{\hookrightarrow} \mathrm{Aut}(G_k).$$

Remark

For NF(number fields),

$$\mathrm{Isom}(\overline{F}_1/F_1, \overline{F}_2/F_2) \xrightarrow{\sim} \mathrm{Isom}(G_{F_1}, G_{F_2}).$$

This is the famous **Neukirch-Uchida Theorem**([NSW]).

A Version of Grothendieck Conjecture

For of MLF type groups, Neukirch-Uchida does not hold. But let's see if an analogy can somehow be drawn. First, in fact:

Lemma

For k_1, k_2 MLF,

$$\mathrm{Isom}(k_1, k_2) \to \mathrm{Out}(G_{k_1}, G_{k_2}) := \mathrm{Isom}(G_{k_1}, G_{k_2}) / \mathrm{Inn}(G_{k_2})$$

is injective.

This follows immediately from local class field theory, but we dare to show it from the **slimness** of groups of MLF type.

Definition

Let G be a topological group. Then, G is **slim** if for any open subgroup H of G, $Z_H(G) := \{g \in G : \forall h \in H, hg = gh\}$ is trivial.

A Version of Grothendieck Conjecture

Lemma

Of MLF type groups are slim.

Proof.

Let $G \simeq G_K$, H be an open subgroup of G and $\gamma \in Z_G(H)$. Then, it can be written as $H \simeq \mathrm{Gal}(\overline{K}/L)$ for some finite Galois extension $K \to L \to \overline{K}$. Now, by local class field theory, the **reciprocity map** fits into the following commutative diagram:

$$\begin{array}{ccc} L^\times & \xrightarrow[\gamma]{\sim} & L^\times \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\overline{K}/L)^{\mathrm{ab}} & \longrightarrow & \mathrm{Gal}(\overline{K}/L)^{\mathrm{ab}} \end{array}$$

In particular, by the fact $\gamma \in Z_G(H)$ and injectivity of the reciprocity map, it follows that γ is "over L" and $\gamma \in \mathrm{Gal}(\overline{K}/L) \simeq G_L$. By letting L vary, it follows that γ is trivial.

□

A Version of Grothendieck Conjecture

Now, injectivity of

$$\mathrm{Isom}(k_1, k_2) \to \mathrm{Out}(G_{k_1}, G_{k_2})$$

in evident by the following argument:

Proof.

Assuming that images of $\phi_1, \phi_2 \in \mathrm{Isom}(k_1, k_2)$ coincide, as ϕ_1, ϕ_2 are over $\mathbb{Q}_p$, it follows that $\phi_1 \circ \phi_2^{-1} \in Z_{G_{\mathbb{Q}_p}}(G_{k_2})$ and we win. □

However.

$$\mathrm{Isom}(k_1, k_2) \to \mathrm{Out}(G_{k_1}, G_{k_2})$$

is not surjective, we would like to consider whether the image of this map can be characterized group-theoretically.

A Version of Grothendieck Conjecture

In fact:

Theorem ([QpGro])

For an outer isomorphism $\alpha : G_{k_1} \xrightarrow{\sim} G_{k_2}$, TFAE:

1. α is **scheme-theoretic**, i.e., it is contained in the image of $\mathrm{Isom}(k_1, k_2) \to \mathrm{Out}(G_{k_1}, G_{k_2})$.
2. α is compatible with the **ramification filtrations**.
3. α preserves **Hodge-Tate representations**.

First, let us discuss Hodge-Tate-ness.

Hodge-Tate Representations

Consider for the model object. Let K be an MFL, V be a finite $\mathbb{Q}_p$-vector space, and $G_K \curvearrowright V$ be a continuous action. Then, let

$$d_V(i) := \dim_{\mathbb{Q}_p}(V(-i) \otimes_{\mathbb{Q}_p} \overline{K}^{\wedge})^{G_K}.$$

Here, $V(i) := V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(i)$ is the **Tate-twist**.

Lemma

$d_V := \sum d_V(i) \leq \dim_{\mathbb{Q}_p}(V)$.

In light of this:

Definition

We shall say V is **Hodge-Tate** if $d_V = \dim_{\mathbb{Q}_p}(V)$ holds.

Reconstruction of Hodge-Tate Representations

This notion is in fact not strictly scheme-theoretic but group-theoretic, i.e., for an of MLF type group G with the ramification filtration,

- $G_K \curvearrowright V(i)$.
- $G_K \curvearrowright \overline{K}^{\wedge}$.

can be reconstructed group theoretically.

The former reconstruction is immediate as we have already seen that $\Lambda(G) \simeq \lim \mu_{\overline{K}}[n]$ can be reconstructed from G. To reconstruct $\overline{K}^{\wedge}$ from G, as we have

$$\overline{K}^{\wedge} \simeq (\mathcal{O}_{\overline{K}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\wedge}$$

it is sufficient to reconstruct $G_K \curvearrowright \mathcal{O}_{\overline{K}}$ (N.B., p can be reconstructed from G). For this reconstruction, we use the ramification filtration $\{G_K^v\}_{v \geq 0}$.

Reconstruction of Hodge-Tate Representations

Reconstruction of $\overline{K}$: first, for an of MLF type G, let

$$U(G) := \mathrm{im}(I(G) \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}})$$

then this reconstructs $U_K = \mathcal{O}_K^{\times}$. By considering **Verlagerung maps** as transition morphisms, $\lim_{H:\mathrm{open}} U(H) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is via p-adic logarithmic isomorphic to the additive module $\overline{K}$.

Reconstruction of $\mathcal{O}_{\overline{K}}$: for an open subgroup $H \subseteq G$ and v, r which satisfies $v = r \cdot e(H),\ 2 \leq r$, as

$$\log : p^{-r} \cdot \mathrm{im}(H^v \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}}) = p^{-r} \cdot U^v(G) \xrightarrow{\sim} \mathcal{O}_L,$$

taking $\lim$ one reconstructs $\mathcal{O}_{\overline{K}}$.

Reconstruction of Hodge-Tate Representations

Thus, we find that Hodge-Tate-ness can be recovered from G. Finally, we observe that $\alpha : G_{k_1} \xrightarrow{\sim} G_{k_2}$ which is compatible with the ramification filtration is scheme theoretic.

The point of the proof is to show the existence of embeddings $\iota_1 : k_1 \to E$ and $\iota_2 : k_2 \to E$ such that

$$\mathrm{im}(\iota_1) = \mathrm{im}(\iota_2)$$

(N.B., this is a strict equality!). That is "to realise k_1, k_2 as set theoretically equal subfields of some field E."

Uniformizing Representations

Definition

Let K be an MLF, E be $\mathbb{Q}_p \to K \to E$ which is finite Galois over $\mathbb{Q}_p$, V be a finite $\mathbb{Q}_p$-vector space which is equipped with a continuous action $G_K \curvearrowright V,\ E \curvearrowright V$ and satisfies $\dim_E(V) = 1$. Then, V is **uniformizing** if there exists an open subgroup $I \subseteq U_K$ such that the restriction of $\rho_K : G_K \to E^\times$ to I is equal to the restriction of some field morphism $K \to E$ to I.

In fact, uniformizing-ness is group theoretic.

Lemma

TFAE:

1. V is uniformizing.
2. $d_V(1) = [E : K] = \dim_{\mathbb{Q}_p}(E)/[K : \mathbb{Q}_p]$ and $d_V(0) = [E : K]([K : \mathbb{Q}_p] - 1)$ hold.

N.B., $[K : \mathbb{Q}_p]$ can be reconstructed from G.

A Version of Grothendieck Conjecture — Main Theorem

Let us show the main theorem. Let $\alpha : G_{k_1} \xrightarrow{\sim} G_{k_2}$ be an outer isomorphism which is compatible with the ramification filtrations.

Take an uniformizing V over k_1. Hence, there exists $I_1 \subseteq U_{k_1}$ and $\iota_1 : k_1 \to E$. V is also uniformizing over k_2, so as we have $I_2 \subseteq U_{k_2}$ and $\iota_2 : k_2 \to E$, and as for any open subgroups $I \subseteq U_K$ generates K as a $\mathbb{Q}_p$-vector space, by calculating commutative diagrams (i.e., as the image of $I_1 \to G_{k_1} \to E$ and $I_2 \to G_{k_2} \to E$ coincide) we obtain

$$\mathrm{im}(\iota_1) = \mathrm{im}(\iota_2)$$

and we win.

Hyperbolic Curves of Strictly Belyi Type

Can one reconstruct K in a mono-anabelian fashion from G?

Definition

A hyperbolic curve X over an MLF K is of **strictly Belyi type** if X is defined over a number field and is isogenous to a hyperbolic curve of genus zero.

Then,

Theorem ([AbsTopIII])

One can reconstruct the quotient and the action

$$\Pi \twoheadrightarrow G \curvearrowright K$$

in a purely group-theoretic fashion from the étale fundamental group Π of X.

Witt Rings

As we saw earlier, p-adic Hodge theory is a fundamental tool of anabelian geometry. Interestingly, however, the so-called Grothendieck conjecture does not hold for **Fargues-Fontaine curves**! We will now see what the point is. For this purpose, we will review basic objets such as Witt vector rings and perfect rings.

We always have a natural operation on arbitrary rings: $R \mapsto R/pR$. For example, when $R = \mathbb{Z}_p$, this is the same as considering

$$\mathbb{Z}_p \mapsto \mathbb{Z}_p/p\mathbb{Z}_p \simeq \mathbb{F}_p.$$

This direction is easy and trivial, then how about the converse direction? Do we have always canonical converse direction? Restricting on perfect $\mathbb{F}_p$-algebras, we always have the converse and it is given by the **Witt vector ring functor**.

Construction of Witt Rings

Let S be a **truncation set**, i.e., $S \subseteq \mathbb{N}_+$ such that

$$\forall n, d \; [n \in S, d | n \implies d \in S].$$

Let R be a ring, and $W_S(R)$ be R^S as a set. Define a map (**ghost map**)

$$w : W_S(R) \to R^S$$

to be the map which satisfies for each $n \in S, (x_i) \in W_S(R)$

$$w_n((x_i)) := \sum_{d|n} d x_d^{n/d}.$$

Construction of Witt Rings

Lemma

There exists an unique, functorial ring structure on W_S such that the ghost maps are ring morphisms.

The proof is that, choosing a Frobenius lifting $\phi_p : R \to R$ for each p, for any $x \in R^S$ we see that

- $x \in \mathrm{im}(w)$ and
- $\forall n \in S\ [1 \leq v_p(n) \implies x_n = \phi_p(x_{n/p}) \mod p^{v_p(n)}R]$

are equivalent (Dwork's Lemma). That is, $\mathrm{im}(w)$ can be characterized by a fixed Frobenius lifting.

Construction of Witt Rings

First, applying Dwork's Lemma to $\mathbb{Z}[X_n, Y_n \mid n \in S]$ and the Frobenius map ϕ_p defined by "raising indeterminates to the p-th power", for $(X_n), (Y_n) \in W_S(\mathbb{Z}[X_n, Y_n \mid n \in S])$, it is immediate that

$$w((X_n)) + w((Y_n)),\ w((X_n))w((Y_n)),\ -w((X_n)) \in \mathrm{im}(w)$$

hold. As $\mathbb{Z}[X_n, Y_n \mid n \in S]$ is torsion free, there exists uniquely $\sigma, \pi, \iota \in W_S(\mathbb{Z}[X_n, Y_n \mid n \in S])$ such that

- $w(\sigma) = w((X_n)) + w((Y_n))$.
- $w(\pi) = w((X_n))w((Y_n))$.
- $w(\iota) = -w((X_n))$.

hold.

Construction of Witt Rings

Next, for a general torsion free ring R, for $x = (x_n), y = (y_n) \in W_S(R)$ there exists uniquely $f : \mathbb{Z}[X_n, Y_n \mid n \in S] \to R$ such that

- $f((X_n)) = x$.
- $f((Y_n)) = y$.

hold. Then, for $W_S(f) : W_S(\mathbb{Z}[X_n, Y_n \mid n \in S]) \to W_S(R)$, defime as follows:

- $x + y := W_S(f)(\sigma)$.
- $xy := W_S(f)(\pi)$.
- $-x := W_S(f)(\iota)$.

This defines a ring structure on $W_S(R)$. For a general ring, choose a "torsion free covering" $R' \twoheadrightarrow R$ and by

$$W_S(R') \twoheadrightarrow W_S(R)$$

one obtains a ring structure on $W_S(R)$ by transporting the ring structure on $W_S(R')$.

An Example

For example, in the case $S_2 := \{1, p\}$, for $(a_1, a_p), (b_1, b_p) \in W_S(R)$, it is immediate to see that

- $(a_1, a_p) \times (b_1, b_p) = (a_1 b_2, a_1^p b_p + a_p b_1^p + p a_p b_p)$.
- $(a_1, a_p) + (b_1, b_p) = (a_1 + b_2, \frac{(a_1^p + b_1^p) - (a_1 + b_1)^p}{p} + (a_p + b_p))$.

Hence, one sees that $\mathbb{Z}/p^2\mathbb{Z} \simeq W_{S_2}(\mathbb{F}_p)$.

What we want to do: let $S_n := \{1, p, \cdots, p^{n-1}\}$, then we have

$$W_{S_n}(\mathbb{F}_p) \simeq \mathbb{Z}/p^n\mathbb{Z}$$

and hence letting $S = \{1, p, p^2, \cdots\}$, we have

$$W(\mathbb{F}_p) := W_S(\mathbb{F}_p) \simeq \lim W_{S_n}(\mathbb{F}_p) \simeq \lim \mathbb{Z}/p^n\mathbb{Z} \simeq \mathbb{Z}_p.$$

Strict p-rings

Recall that there exists an unique multiplicative section $\mathbb{F}_p \xrightarrow{\tau} \mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$ (**Teichmüller representative**). Review of the construction: for $x \in \mathbb{F}_p \simeq \mathbb{Z}_p/p$ and $n \geq 1$, as $\mathbb{F}_p$ is perfect, there exists

$$x^{p^{-n}}$$

Once one takes lifts $y_n \in \mathbb{Z}_p$, it is immediate that these form a Cauchy sequence, and

$$\lim y_n \in \mathbb{Z}_p$$

exists and is independent of the choice of lifts. Let $\tau(x) := \lim y_n$.

Now,

$$W(\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Z}_p, \ (x_n) \mapsto \sum \tau(x_n) p^n.$$

Strict p-rings

This construction may be generalized to general perfect rings. For a perfect ring R, $W(R)$ can be intrinsically characterized as a strict p-ring $(R \mapsto R/p,\ R' \mapsto W(R'))$:

Definition

A ring R is a **strict p-ring** if the followings hold:

1. R is p-adically separated and complete.
2. R/p is perfect.
3. $R \stackrel{\times p}{\to} R$ is injective.

N.B., these conditions are purely ring-theoretic.

δ-rings

Definition

A pair (R, δ) is a δ-**ring** if R is a ring and $\delta : R \to R$ is a map such that

1. $\delta(0) = \delta(1) = 0$.
2. $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.
3. $\delta(x + y) = \delta(x) + \delta(y) + \frac{(x^p+y^p)-(x+y)^p}{p}$.

hold.

For a δ-ring (R, δ) one may construct a Frobenius lifting

$$\phi : R \to R,\ f \mapsto f^p + p\delta(f).$$

Lemma

If R is p-torsion free, this construction gives rise to a bijection between the two sets below;

1. The set of δ-structure on R.
2. The set of Frobenius liftings on R.

Perfect δ-rings

We shall say that a δ-ring (R, δ) is perfect if the Frobenius lifting ϕ determined by δ is an isomorphism. Then,

Lemma

There exists a category equivalence induced by $R \mapsto W(R)$:

1. The category of perfect rings R which are of characteristic p.
2. The category of classically p-adically complete perfect δ-rings (R, δ).

Proof.

Let us show $(1) \implies (2)$. We want to lift the Frobenius on R to $W(R)$, but as $\lim W(R)/p^n \simeq W(R)$, it is sufficient to lift it to $W(R)/p^n$. By the way, the Frobenius on R induces an automorphism on $L_{R/\mathbb{F}_p}$, and by the fact that $d(x^p) = 0$ it is immediate that it is a zero map, so we get $L_{R/\mathbb{F}_p} = 0$. By deformation theory, we win. □

Distinguished Elements

Definition

Let (R, δ) be a δ-ring. Then, we shall say that $d \in R$ is **distinguished** if $\delta(d) \in R^\times$.

Lemma

Let R be a perfect $\mathbb{F}_p$-algebra and $d \in W(R)$ be a distinguished element. Then, d is a non-zero divisor.

Let R be a ring, $R^\flat := \lim R/p$ be its **tilt**. The canonical morphism $R^\flat \to R/p$ has, by the same argument before, a lift

$$\begin{array}{ccc} \mathbb{A}_{\mathsf{inf}}(R) := W(R^\flat) & \longrightarrow & R \\ \big\downarrow & & \big\downarrow \\ R^\flat & \longrightarrow & R/p. \end{array}$$

We shall call this lift the **Fontaine's map** θ_R.

Perfect Prisms

Definition ([BS19])

We shall say that $((R, \delta), I)$ is a **prism** if the followings hold:

1. (R, δ) is a δ-ring,
2. $I \subseteq R$ is an ideal, $V(I) \subseteq \mathrm{Spec}(R)$ defines a **Cartier divisor**,
3. A is (p, I)-derived compelete, and
4. $p \in (I, \phi(I))$.

A perfect prism is a prism such that (R, δ) is perfect.

Now first, let us give a model-explicit definition of perfectoid rings.

Definition

R is a perfectoid ring if there exists a perfect prism $((S, \delta), I)$ such that $R = S/I$.

Perfectoid Rings

Perfect rings can be defined model-implicitly.

Theorem

Let R be a ring. Then, being R is a perfectoid is equivalent to that the following four holds:

1. R is classically p-adically complete.
2. The Frobenius morphism $R/p \to R/p$ is surjective.
3. The kernel of Fontaine's map $\theta_R : \mathbb{A}_{\mathsf{inf}}(R) \to R$ is principal.
4. There exists some $\varpi \in R$ and $u \in R^\times$ such that $\varpi^p = pu$.

We shall show that $(\mathbb{A}_{\mathsf{inf}}(R), \ker(\theta_R))$ is the desired perfect prism. First, take a generator d of $\ker(\theta_R)$, and one can show that d is distinguished. By the previous lemma, we see that $\mathbb{A}_{\mathsf{inf}}(R)$ is a classically p-adically complete δ-ring. That $\ker(\theta_R)$ defines a Cartier divisor is clear by the fact that d is a non zero-divisor.

Perfectoid Rings

To see that it is derived $(p, \ker(\theta_R))$-complete, it suffices to see that it is classically $(p, \ker(\theta_R))$-complete, and it is achieved by seeing that $\mathbb{A}_{\mathsf{inf}}(R)/p$ is d-complete. This follows from the fact that for $\forall y \in \ker(R^\flat \to R/p)$, $R^\flat$ is y-complete.

$p \in (\ker(\theta_R), \phi(\ker(\theta_R)))$is clear. Finally, we shall see that

$$\theta_R : \mathbb{A}_{\mathsf{inf}}(R)/\ker(\theta_R) \xrightarrow{\sim} R,$$

but this is clear once one recalls that θ_R is a lift of $R^\flat \to R/p$ hence is surjective.

Definition

A topological field K is a **perfectoid field** if K is a perfectoid ring and its topology is induced from a rank 1 valuation $K \to \mathbb{R}_{\geq 0}$.

Fargues-Fontaine curves

Theorem ([KJ])

Let F be an algebraically closed perfectoid field of characteristic p, K_1, K_2 be MLF, $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$ a strictly scheme theoretic (i.e., non-group theoretic) group isomorphism. Then, Fargues-Fontaine curves X_{F,K_1}, X_{F,K_2} are not mutually isomorphic.

The key point of the proof is that:

- There exists a **crystalline representation** on G_1 such that after composing with the group theoretic isomorphism $\alpha : G_{K_1} \xrightarrow{\sim} G_{K_2}$, it, a representation on G_2, is not Hodge-Tate.
- Any scheme theoretic isomorphisms $X_{F,K_1} \xrightarrow{\sim} X_{F,K_2}$ preserves crystalline representations.

Let K^{unr} be the maximal unramified subextension. Recall that V is crystalline if and only if $\dim_{\mathbb{Q}_p}(V) = \dim_{K^{\mathsf{unr}}}(D_{\mathsf{cris}}(V))$ holds and that K^{unr} is group theoretic, i.e., $G_{K_1} \xrightarrow{\sim} G_{K_2}$ reconstructs $K_1^{\mathsf{unr}} \xrightarrow{\sim} K_2^{\mathsf{unr}}$.

Now, assume that there exists a scheme theoretic isomorphism $\phi : X_{F,K_2} \xrightarrow{\sim} X_{F,K_1}$ by considering the pull-back, we have

$$H^0(X_{F,K_1}, V \otimes \mathcal{O}_{X_{F,K_1}}) \simeq H^0(X_{F,K_2}, \phi^*(V \otimes \mathcal{O}_{X_{F,K_1}})).$$

As $D_{\mathsf{cris}}(V) \otimes_{K^{\mathsf{unr}}} K \simeq H^0(X_{F,K}, V \otimes \mathcal{O}_{X_{F,K}})$ holds, we also have

$$\dim_{K_1}(D_{\mathsf{cris}}(V) \otimes_{K_1^{\mathsf{unr}}} K_1) = \dim_{K_2}(D_{\mathsf{cris}}(V) \otimes_{K_2^{\mathsf{unr}}} K_2).$$

Fargues-Fontaine curves

Hence, we have

$$\dim_{K_1^{\mathsf{unr}}}(D_{\mathsf{cris}}(V)) \cdot [K_1 : K_1^{\mathsf{unr}}] = \dim_{K_2^{\mathsf{unr}}}(D_{\mathsf{cris}}(V)) \cdot [K_2 : K_2^{\mathsf{unr}}]$$

but as $[K : K^{\mathsf{unr}}]$ is group-theoretic,

$$\dim_{K_1^{\mathsf{unr}}}(D_{\mathsf{cris}}(V)) = \dim_{K_2^{\mathsf{unr}}}(D_{\mathsf{cris}}(V))$$

holds. Finally we have

$$\dim_{K_1^{\mathsf{unr}}}(D_{\mathsf{cris}}(V)) = \dim_{\mathbb{Q}_p}(V) = \dim_{K_2^{\mathsf{unr}}}(D_{\mathsf{cris}}(V))$$

and this leads to contradiction.

Radial Environments

Definition ([IUTeichII])

1. A triple $(\mathbf{R}, \mathbf{C}, \Phi)$ is a **radial environment** if $\mathbf{R}, \mathbf{C}$ are connected groupoids and $\Phi : \mathbf{R} \to \mathbf{C}$ is an essentially surjective functor.
2. A radial environment in which Φ is full is called **multiradial**, otherwise **uniradial**.

Multiradial environment is "stable under switching" :

Lemma

The functor defined by "switching"

$$\mathbf{R} \times_{\mathbf{C}} \mathbf{R} \to \mathbf{R} \times_{\mathbf{C}} \mathbf{R}$$

preserves isomorphism classes of objects.

Tautological Resolution of Indeterminacies

The most basic example: let $\mathbf{R}$ be the category of one-dimensional complex vector spaces, i.e., $\mathbb{C}$ modules M such that there exists a $\mathbb{C}$-isomorphism $M \simeq \mathbb{C}$, $\mathbf{C}$ be the category of two-dimensional $\mathbb{R}$-modules, and $\Phi : \mathbf{R} \to \mathbf{C}$ be the "forgetful functor". In this case, $(\mathbf{R}, \mathbf{C}, \Phi)$ is uniradial as

$$\mathbb{C}^\times \simeq \mathrm{GL}_1(\mathbb{C}) \overset{\text{non-isom}}{\hookrightarrow} \mathrm{GL}_2(\mathbb{R}).$$

However, once we replace the objects of $\mathbf{R}$ by pairs of M and a $\mathrm{GL}_2(\mathbb{R})$-orbit of some $\mathbb{C}$-isomorphism $M \simeq \mathbb{C}$, $(\mathbf{R}, \mathbf{C}, \Phi)$ is multiradial.

Tautological Resolution of Indeterminacies

Another example comes from of MLF-type groups.

Recall that from the étale fundamental group Π of a of strictly Belyi type hyperbolic curve X over an MLF K we can reconstruct the quotient G and $G \curvearrowright K$. Let us review the reconstruction algorithm of the quotient G:

Tautological Resolution of Indeterminacies

1. To reconstruct the quotient G, it suffices to reconstruct the subobject Δ, i.e., the geometric fundamental group of X.
2. First, as we reconstructed p from G, we can reconstruct p from Π as follows; let p be the unique prime such that for infinitely many ℓ

$$\dim_{\mathbb{Q}_p}(\Pi^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) - \dim_{\mathbb{Q}_\ell}(\Pi^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_\ell) \neq 0$$

   holds.
3. Reconstruct $\Delta \subseteq \Pi$ as the intersection of all open subgroups $\Pi' \subseteq \Pi$ such that for all $\ell \neq p$

$$\dim_{\mathbb{Q}_p}(\Pi'^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) - \dim_{\mathbb{Q}_\ell}(\Pi'^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_\ell)$$

$$= [\Pi : \Pi'](\dim_{\mathbb{Q}_p}(\Pi^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) - \dim_{\mathbb{Q}_\ell}(\Pi^{\mathsf{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_\ell))$$

   holds.

Tautological Resolution of Indeterminacies

This reconstruction algorithm gives rise to a functor

$$\mathbf{R} \to \mathbf{C},\ \Pi \mapsto G$$

after defining categories as follows:

- Let $\mathbf{R}$ be the category whose objects are topological groups which are isomorphic to Π.
- Let $\mathbf{C}$ be the category whose objects are topological groups which are isomorphic to G.

This functor is, by the existence of strictly scheme theoretic isomorphisms $K \xrightarrow{\sim} K$, an uniradial environment.

Also in this example, after substituting objects of $\mathbf{R}$ by pairs of Π and the $\mathrm{Aut}(G_K)$-orbit of the reconstructed reference isomorphism $G(\Pi) \simeq G_K$, we get a multiradialized environment.

Rigidity Implies Multiradiality

This example suggests a way to give a general radial environment its **tautological multiradialization**.

Definition

Define the tautological multiradialization $(\mathbf{R}^{\text{mtz}}, \mathbf{C}, \Phi^{\text{mtz}})$ of $(\mathbf{R}, \mathbf{C}, \Phi)$ as follows:

1. Define $\mathbf{R}^{\text{mtz}}$ to be $\mathbf{R}$ whose objects are substituted by $(R, C, \alpha : \Phi(R) \overset{\text{full poly}}{\overset{\sim}{\to}} C)$.
2. $\Phi^{\text{mtz}}((R, C, \alpha)) := C$.

It is immediate that tautological multiradializations are multiradial. When the full poly-isomorphisms above α are mono-isomorphisms, $(\mathbf{R}, \mathbf{C}, \Phi)$ turns to be multiradial. This is the same as C is being rigid.

Rigidity Implies Uniqueness of Reference

As we saw at the beginning of this talk, “indeterminacy due to different choices of reference isomorphisms” appears frequently in mathematics, and its resolution also appears frequently. Fix an object C and define an object to be of C type if it is in some sense isomorphic to C, then;

Lemma

C is rigid if and only if for any of C-type object D its reference isomorphism is unique.

Rigidity of $\mathbf{AbGrp}$

Rigidity revisited: a category $\mathbf{C}$ is called **rigid** if $\mathrm{Aut}(\mathbf{C})$ is trivial, i.e., every auto-category equivalence $\mathbf{C} \xrightarrow{\sim} \mathbf{C}$ is isomorphic to id.

Usual "algebraic" categories are rigid. For example, the category of abelian groups $\mathbf{AbGrp}$ is rigid by the argument of Freyd which asserts category theoretic characterizations of underlying sets and additive operations on them([Fre]):

1. $\mathbb{Z}$ has the following categorical characterization:
   1.1 For every $G \in \mathbf{AbGrp}$ which is not initial, $|\mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, G)| > 1$.
   1.2 For every $f \in \mathrm{End}_{\mathbf{AbGrp}}(\mathbb{Z})$ such that $f \circ f = f$, either $f = \mathrm{id}$ or $f = 0$.
2. $\mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, G) \simeq G$ as sets.
3. For $g, h \in \mathrm{Hom}_{\mathbf{AbGrp}}(\mathbb{Z}, G) \simeq G$, $g + h$ is obtained by considering the universal property of coproduct $\mathbb{Z} \times \mathbb{Z}$.

As any category equivalence preserves these category theoretic properties, it must be isomorphic to identity functor.

Rigidity of $\mathbf{Sets}$

By the Grothendieck conjecture for connected anabelioids, i.e.,

$$\mathrm{Isom}(G_1, G_2) \xrightarrow{\sim} \mathrm{Isom}(\mathbf{FSets}_{G_1}, \mathbf{FSets}_{G_2})$$

one can deduce the fact that $\mathbf{FSets}$ is rigid. Similar arguments yields the fact that $\mathbf{Sets}$ is also rigid.

Categorical Reconstruction of Schemes

Let X be a locally noetherian scheme and $\mathbf{Sch}(X)$ be the category whose objects are $Y \to X$ which is of finite type and morphisms are $Y_1 \to Y_2$ which is of finite type over X. Then,

Theorem ([LocN])

From $\mathbf{Sch}(X)$ we can reconstruct X purely category theoretically.

To reconstruct the scheme X, we have to reconstruct

1. its underlying topological space: this can be reconstructed, as $|X|$ is sober, by the famous reconstruction algorithm, from $\mathbf{Shv}(X)$. To reconstruct $\mathbf{Shv}(X)$, it suffices to reconstruct open immersions on X and coverings on X.
2. its structure ring sheaf: it suffices, as $\mathcal{O}_X(U) \simeq \mathrm{Hom}(U, \mathbb{A}^1_X)$, to reconstruct $\mathbb{A}^1_Y$ and this can be done by reconstructing $\mathbb{P}^1_Y$.

Categorical Reconstruction of Smoothness

Let us reconstruct open immersions. Being $Y_1 \to Y_2$ is an open immersion is equivalent to it being a smooth monomorphism. Hence it suffices to reconstruct smoothness. But this follows from the famous characterization, which one can find in EGA, below:

Lemma

Being $Y_1 \to Y_2$ smooth is equivalent to that for any monomorphism

$$Z_0 \hookrightarrow Z$$

from an one-pointed scheme and $Z_0 \to Y_1, Z \to Y_2$ such that

$$\begin{array}{ccc} Z_0 & \longrightarrow & Y_1 \\ \big\downarrow & & \big\downarrow \\ Z & \longrightarrow & Y_2 \end{array}$$

commutes, there exists a compatible $Z \to Y_1$.

Categorical Reconstruction of Open Coverings

It is immediate to reconstruct one-pointed schemes:

Lemma

1. Being $Y \to X$ a reduced one pointed scheme and, as an object in $\mathbf{Sch}(X)$, being $Y \to X$ is minimal is equivalent.
2. Being $Y \to X$ an one-pointed scheme and that there exists, up-to isomorphism, an unique morphism $Z \to Y$ from a reduced one pointed scheme.

From this, one can reconstruct open coverings:

Lemma

A family of objects $\{U_i \to X\}$ is an open covering if and only if $U_i \to X$ are open immersions and any reduced one-pointed monomorphism $Z \to X$ factors through some U_i.

Categorical Reconstruction of Topological Spaces

Review: let us review how to reconstruct a sober space X from $\mathbf{Shv}(X)$ ([Top]). Let $\mathbf{C}$ be a category which is equivalent to $\mathbf{Shv}(X)$. Then,

1. Define $\mathcal{O}^T(\mathbf{C})$ to be the set of subobjects of the terminal object of $\mathbf{C}$ modulo isomorphism.
2. $X(\mathbf{C}) := \mathrm{Hom}_{\mathsf{frame}}(\mathcal{O}^T(\mathbf{C}), *)$.
3. For $A \in \mathcal{O}^T(\mathbf{C})$, $U_A := \{\phi \in X(\mathbf{C}) : \phi(A) = *\}$.
4. $\mathcal{O}(\mathbf{C}) := \{U_A : A \in \mathcal{O}^T(\mathbf{C})\}$.

Then, by choosing a reference isomorphism, we have $(X, \mathcal{O}) \simeq (X(\mathbf{C}), \mathcal{O}(\mathbf{C}))$. Thus we reconstructed $(X, \mathcal{O})$ from $\mathbf{C}$ in a purely categorical fashion.

Rigidity of $\mathbf{Rng}$

Stronger theorem, for general schemes X, Y, one can show that

$$\mathrm{Isom}(X, Y) \xrightarrow{\sim} \mathrm{Isom}(\mathbf{Sch}_Y, \mathbf{Sch}_X).$$

From this stronger theorem, one immediately deduces that

$$\mathrm{Isom}(R_1, R_2) \xrightarrow{\sim} \mathrm{Isom}(\mathbf{Alg}_{R_1}, \mathbf{Alg}_{R_2}).$$

In particular letting $R_1 = R_2 = \mathbb{Z}$,

Theorem ([AutCS])

$\mathbf{Rng}$ is rigid.

Galois Categories

A category $\mathbf{C}$ is **Galois** if there exists a **profinite group** G and category equivalence $\mathbf{C} \simeq \mathbf{FSets}_G$. This definition is **model-explicit**. However, we have a **model-implicit** characterization of Galois categories: let $\mathbf{C}$ be a category, then $\mathbf{C}$ is Galois if and only if the followings hold:

1. $\mathbf{C}$ is finite complete and finite co-complete.
2. $\mathbf{C}$ does not have the zero object.
3. Every morphism of $\mathbf{C}$ has a strong-epi/mono factorization.
4. There exists a functor $\mathbf{C} \to \mathbf{FSets}$ which is conservative and exact.

The functor that appears in the last fourth axiom is called a **fiber functor** (N.B., fiber functors are mutually isomorphic). For a Galois category $\mathbf{C}$ and a fiber functor $\mathcal{F}$, define

$$\pi_1(\mathbf{C}, \mathcal{F}) := \mathrm{Aut}(\mathcal{F})$$

and we call it the **fundamental group** of $\mathbf{C}$.

Strong-epi/Mono Factorization

A **strong epi/mono factorization** of a morphism $u : X \to Y$ is a quintuple

$$(u_1, u_2, v, Z_1, Z_2)$$

where $u_1 : X \to Z_1$ is a strong epimorphism, $u_2 : Z_1 \to Y$ is a monomorphism, $v : Z_1 \coprod Z_2 \xrightarrow{\sim} Y$ is an isomorphism such that $u = u_2 \circ u_1$ and

$$\begin{array}{ccccc} X & \xrightarrow{u_1} & Z_1 & \xrightarrow{\quad u_2 \quad} & Y \\ & & & \searrow & \uparrow{\scriptstyle v}\,\wr \\ & & & & Z_1 \coprod Z_2 \end{array}$$

commutes. This is a categorical analogy of "factorization into an injection after a surjection".

Strong Epimorphisms

Let us review the notion of strong epimorphisms.

Definition

For a morphism $f : X \to Y$ and an object A, let

$$\mathrm{Hom}_{\mathbf{C}}(X, A)^{\dagger}_{Y} := \{u \in \mathrm{Hom}_{\mathbf{C}}(X, A) : u \circ p_1 = u \circ p_2\}.$$

Here, p_i are the canonical projection $X \times_Y X \to X$ associated to the fiber product.

Definition

We shall say that a morphism $f : X \to Y$ is a **strong epimorphism** if it is epimorphic and for any object A, the natural map

$$\mathrm{Hom}_{\mathbf{C}}(X, A) \to \mathrm{Hom}_{\mathbf{C}}(X, A)^{\dagger}_{Y}$$

is a bijection.

Stability of Strong Epimorphisms

Remark

In general categories, strong epimorphisms are NOT stable under compositions([KS]).

When one speaks of Galois categories, strong epimorphisms are always stable under compositions.

Proof.

It is immediate that f being a strong epimorphism and $\mathcal{F}(f)$ being surjective are equivalent by the existence of strong epi-mono factorizations. Then, as surjections are stable under composition. it is clear. □

Fundamental Groups

Fundamental groups are profinite:

Proof.

There exists a natural inclusion $\mathrm{Aut}(\mathcal{F}) \hookrightarrow \prod \mathrm{Aut}_{\mathbf{C}}(X)$ which realizes $\mathrm{Aut}(\mathcal{F})$ as a closed subgroup of $\prod \mathrm{Aut}_{\mathbf{C}}(X)$. It is sufficient to see that it is **totally disconnected**, **quasi-compact** and **Hausdorff**. As each $\mathrm{Aut}_{\mathbf{C}}(X)$ has the discrete topology, $\prod \mathrm{Aut}_{\mathbf{C}}(X)$ satisfies properties above, so considering various elementary stability properties and that the fact $\mathrm{Aut}(\mathcal{F}) \hookrightarrow \prod \mathrm{Aut}_{\mathbf{C}}(X)$ is closed, we win. □

For a general locally Noetherian connected scheme X, finite étale coverings over X form a category

$$\mathbf{F\acute{e}t}(X)$$

and it is Galois with a fiber functor

$$\mathcal{F}_{\overline{s}} : \mathbf{F\acute{e}t}(X) \to \mathbf{FSets}, \ (Y \to X) \mapsto \left|\mathrm{Hom}_{\mathbf{F\acute{e}t}(X)}(\mathrm{Spec}(\Omega), Y)\right|$$

defined by a fixed geometric point $\overline{s} : \mathrm{Spec}(\Omega) \to X$.

In particular, if one considers when $X = \mathrm{Spec}(k)$ by restricting **Yoneda Lemma** to isomorphisms, it is immediate that

$$\mathrm{Aut}(\mathcal{F}_{\overline{s}}) \simeq \mathrm{Aut}(\mathrm{Spec}(\Omega)) \simeq \mathrm{Aut}_k(\Omega) \simeq G_k$$

holds. That is, the notion of étale fundamental groups is a generalization of the notion of absolute groups.

Morphisms between Galois Categories

Let us first define an appropriate notion of morphisms between Galois categories.

Definition

A morphism between Galois categories $\phi : \mathbf{C}_1 \to \mathbf{C}_2$ is definend to be an exact functor $\phi^* : \mathbf{C}_2 \to \mathbf{C}_1$.

By this definition, a fiber functor can be seen as a basepoint $\mathbf{FSets} \to \mathbf{C}$ (the fact that it is conservative follows from strong epi/mono factorization).

Morphisms between Galois Categories

In the category of Galois categories, $\mathbf{FSets}$ is just like the "singleton" , i.e., the terminal object. Let me explain this.

Definition ([Anab])

A category $\mathbf{C}$ is an **anabelioid** if there exists a finite family $\{\mathbf{C}_i\}_I$ of Galois categories and a category equivalence $\mathbf{C} \simeq \prod_I \mathbf{C}_i$. An anabelioid $\mathbf{C}$ is **connected** if in particular one can take a model which satisfies $I = *$. Morphisms between anabelioids are defined to be exact functors in the opposite direction.

That is, connected anabelioids are the same as Galois categories.

Canonical Decomposition of Anabelioids

At first glance, this definition may be seen ill-defined, but this is not the case.

Lemma

For an anabelioid $\mathbf{C}$, its connected components can be reconstructed in a purely category theoretic manner.

Proof.

For an anabelioid $\mathbf{C}$, let T be its terminal object, $\coprod_I T_i \simeq T$ be its maximal decomposition, then we have a natural morphism

$$\mathbf{C} \xrightarrow{\sim} \prod_I \mathbf{C}_{T_i}$$

and it is immediate that is gives a category equivalence, so we win. □

Locality of Basepoints

By applying the same argument, one can show a sort of "purity" of morphisms of anabelioids.

Lemma

Let $\phi : \mathbf{C}_1 \to \mathbf{C}_2$ be an anabelioid morphism. Then, ϕ can be canonically written as a morphism obtained by glueing morphisms of their connected anabelioid components.

In particular,

Corollary

For an anabelioid $\mathbf{C}$, its basepoint $\mathbf{FSets} \to \mathbf{C}$ is local, i.e., it factors through some connected anabelioid component $\mathbf{FSets} \to \mathbf{D} \to \mathbf{C}$.

Connected Objects

There are two important classes of objects.

Definition

Let $\mathbf{C}$ be a Galois category.

1. We shall say that $X \in \mathrm{ob}(\mathbf{C})$ is **connected** if there exists objects X_1, X_2 and an isomorphism $X \simeq X_1 \coprod X_2$ then either X_1 or X_2 is initial.
2. A **decomposition** of X is a pair $(\{X_i\}_I, \coprod_I X_i \simeq X)$ such that X_i are connected.
3. We shall say that Y is a connected component of X if there exists a decomposition $(\{X_i\}_I, \coprod_I X_i \simeq X)$ such that for some $i \in I$, $Y = X_i$ holds.

For example, in $\mathbf{FSets}_G$, connected objects are the same as finite sets X whose equipped group action $G \curvearrowright X$ is transitive. Such objects are, under indeterminacy of a choice of a basepoint, isomorphic to G/H for some open subgroup $H \subseteq G$.

Galois Objects

Next, let us define Galois objects.

Definition

Let $\mathbf{C}$ be a Galois category. Then a non-initial connected object $X \in \mathrm{ob}(\mathbf{C})$ is **Galois** if the categorical quotient $X/\operatorname{Aut}_{\mathbf{C}}(X)$ is terminal.

For example, in $\mathbf{FSets}_G$, Galois objects are the same as G/H for some open normal subgroup $H \subseteq G$ canonical up to a choice of a basepoint.

Associated Categories

For a given pointed Galois category $(\mathbf{C}, \mathcal{F})$ one can construct an associated category of pointed objects.

Definition

For a given pointed Galois category $(\mathbf{C}, \mathcal{F})$, let $\mathbf{C}_{\mathcal{F}}^{\mathsf{pt}}$ be the category whose objects are pairs (X, x) such that $x \in \mathcal{F}(X)$ and whose morphisms are morphism in $\mathbf{C}$ which are compatible with the choice of points.

For a given Galois category $\mathbf{C}$ and its Galois object X, one can construct another associated category that is "the subcategory whose objects are under control of X".

Definition

For a given Galois category $\mathbf{C}$ and its Galois object X_0, let $\mathbf{C}^X$ be the full subcategory of $\mathbf{C}$ whose objects are $Y \in \mathrm{ob}(\mathbf{C})$ such that for any Z which is a connected component of Y, $\mathrm{Hom}_{\mathbf{C}}(X, Z) \neq \varnothing$ holds.

Main Theorem

Sketch of the proof of the main theorem: we shall show for a pointed Galois category $(\mathbf{C}, \mathcal{F})$,

$$\mathcal{F} : \mathbf{C} \xrightarrow{\sim} \mathbf{FSets}_{\pi_1(\mathbf{C},\mathcal{F})}$$

holds. First, we show that its basepoint factors as follows:

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{FSets}_{\pi_1(\mathbf{C},\mathcal{F})} \\ & \searrow & \downarrow \\ & & \mathbf{FSets} \end{array}$$

and this follows from the facts below: for a fixed $X \in \mathrm{ob}(\mathbf{C})$,

- There exists a non-initial connected object Y over X and the Galois closure $\widehat{Y} \to Y \to X$ of Y.
- Each object in the essential image of $\mathcal{F}|_G : \mathbf{C}^G \to \mathbf{FSets}$ has a natural $\mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}$-action, and $\mathbf{C}^G \xrightarrow{\sim} \mathbf{FSets}_{\mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}}$.
- $\pi_1(\mathbf{C}, \mathcal{F}) \simeq \lim_{G \in \mathrm{Gal}_{\mathbf{C}}} \mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}$.

Here $\mathrm{Gal}_{\mathbf{C}}$ stands for the isomorphism class of Galois objects in $\mathbf{C}$.

Main Theorem

The strategy to show essential surjectivity:

- For $E \in \mathrm{ob}(\mathbf{FSets}_{\pi_1(\mathbf{C},\mathcal{F})})$ its equipped group action $\pi_1(\mathbf{C},\mathcal{F}) \curvearrowright E$ factors as $\pi_1(\mathbf{C},\mathcal{F})/H \curvearrowright E$ for some open normal subgroup H.
- For a sufficiently large Galois object X corresponding to $\pi_1(\mathbf{C},\mathcal{F})/H$, one sees that

$$\mathbf{C}^X \xrightarrow{\sim} \mathbf{FSets}_{\pi_1(\mathbf{C},\mathcal{F})/H}$$

and we win. For fully faithfulness, once one fixes X, Y, they turn to be under some Galois object G, and for such a G, it follows that $\mathbf{C}^G \xrightarrow{\sim} \mathbf{FSets}_{\mathrm{Aut}_{\mathbf{C}}(G)^{\mathrm{op}}}$ and,

$$\mathrm{Hom}_{\mathbf{C}}(X,Y) = \mathrm{Hom}_{\mathbf{C}^G}(X,Y)$$

$$\simeq \mathrm{Hom}_{\mathrm{Aut}_{\mathbf{C}}(G)^{\mathrm{op}}}(\mathcal{F}(X),\mathcal{F}(Y)) = \mathrm{Hom}_{\mathbf{FSets}_{\pi_1(\mathbf{C},\mathcal{F})}}(\mathcal{F}(X),\mathcal{F}(Y))$$

hold, hence we win.

Main Theorem

Thus, the key points are:

1. that for a not necessarily connected, non-initial object X, there exists a non-initial connected covering $Y \to X$.
2. that for a non-initial connected object X, there exists its "Galois closure" $\widehat{X} \to X$.
3. that there exists a profinite group isomorphism

$$\pi_1(\mathbf{C}, \mathcal{F}) \simeq \lim_{G \in \mathrm{Gal}_{\mathbf{C}}} \mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}.$$

4. that for a fixed Galois object G, each objects in the essential image of the functor $\mathbf{C}^G \to \mathbf{FSets}$ has a natural $\mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}$-action and the fixed basepoint gives rise to a category equivalence

$$\mathbf{C}^G \xrightarrow{\sim} \mathbf{FSets}_{\mathrm{Aut}_{\mathbf{C}}(G)^{\mathsf{op}}}.$$

Connected Closure

The existence of connected closures follows from arguments in $\mathbf{C}_{\mathcal{F}}^{\mathsf{pt}}$.

For a non-initial X, as $\mathcal{F}(X)$ is non-initial, there exists some element $x \in \mathcal{F}(X)$. Then, by taking a decomposition $(\{X_i\}_I, X \simeq \coprod_I X_i)$ of X, for some i, $x \in \mathcal{F}(X_i)$ holds. For this i, $X_i \to X$ gives us the desired connected closure.

The existence of decompositions, follows from the existence of strong epi/mono factorizations and induction.

Remark

As, in Galois categories, every object has an unique connected component decomposition, recalling that in $\mathbf{FSets}_G$ connected objects are the same as G/H for some open subgroup $H \subseteq G$, it follows immediately that in Galois categories, every object has a form

$$G/H_1 \coprod G/H_2 \coprod \cdots \coprod G/H_n.$$

That is, Galois categories are **combinatorial**.

Galois Closure

For a connected object X, by the arguments in $\mathbf{C}_{\mathcal{F}}^{\mathsf{pt}}$, one immediately sees that the existence of (X_0, ζ_0) such that

$$\mathrm{ev}_{\zeta_0} : \mathrm{Hom}(X_0, X) \xrightarrow{\sim} \mathcal{F}(X)$$

holds. Indeed, writing $\mathcal{F}(X) = \{\zeta_1, \cdots, \zeta_n\}$, as $(X, \zeta_1), \cdots, (X, \zeta_n) \in \mathrm{ob}(\mathbf{C}_{\mathcal{F}}^{\mathsf{pt}})$ and

$$\mathcal{F}(X^n) \simeq \mathcal{F}(X)^n$$

hold, one may take $\zeta \in \mathcal{F}(X^n)$ that corresponds to (ζ_i). If it is necessary take a lift of this element to the connected closure.

Writing $\mathrm{Hom}(X_0, X) = \{u_1, \cdots, u_n\}$, by the universality of products, there exists an unique

$$\pi : X_0 \to X^n$$

such that $p_i \circ \pi = u_i$ holds. By taking a factorization of π one obtains

$$X_0 \to \widehat{X} \to X^n$$

and finally this is the desired Galois closure.

Galois System

On the isomorphism class $\mathrm{Gal}_{\mathbf{C}}$ define an order as

$$X \leq Y \iff \exists Y \to X.$$

Then, by the existence of Galois closures, one finds that $(\mathrm{Gal}_{\mathbf{C}}, \leq)$ turns to be a directed partially ordered set. Now, for a fixed $\underline{\zeta} \in \prod_{X \in \mathrm{Gal}_{\mathbf{C}}} \mathcal{F}(X)$, one obtains a unique projective system

$$\mathrm{Gal}_{\mathbf{C}}^{\underline{\zeta}} := (\phi_{X,Y}^{\underline{\zeta}} : Y \to X)$$

such that $\phi_{X,Y}^{\underline{\zeta}}$ preserves the fixed basepoints.

Galois System

This projective system lifts to Aut, i.e., for $\omega_Y \in \mathrm{Aut}(Y)$ there exists an unique $\omega_X \in \mathrm{Aut}(X)$ such that

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow{\scriptstyle \omega_Y} & & \vdots{\scriptstyle \omega_X} \\ Y & \longrightarrow & X \end{array}$$

commutes. This fact is achieved by seeing that

$$\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Hom}(Y, X)$$

with some set theoretic arguments. Therefore, one obtains a group morphism

$$r^{\zeta}_{X,Y} : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)$$

and it is immediate to see that this forms a projective system

$$R^{\zeta}_{\mathbf{C}} : (r^{\zeta}_{X,Y} : \mathrm{Aut}(Y) \to \mathrm{Aut}(X)).$$

Galois System

Finally, it is immediate by some trivial computations to see that

$$u^{\underline{\zeta}} : \pi_1(\mathbf{C}, \mathcal{F}) \xrightarrow{\sim} (\lim_{X \in R^{\zeta}_{\mathbf{C}}} \mathrm{Aut}(X))^{\mathsf{op}}$$

$$\theta \mapsto \mathrm{ev}^{-1}_{\zeta_X}(\theta_X(\zeta_X)).$$

Here $\mathrm{ev}_{\zeta,X}$ is the bijection

$$\mathrm{ev}_{\zeta,X} : \mathrm{Hom}(X, X) \xrightarrow{\sim} \mathcal{F}(X).$$

Grothendieck Conjecture for Connected Anabelioids

Theorem

Let G_1, G_2 be profinite groups, $\beta_i : \mathbf{FSets} \to \mathbf{FSets}_{G_i}$ be the tautological basepoints. Then the natural map

$$\mathrm{Isom}(G_1, G_2) \xrightarrow{\sim} \mathrm{Isom}((\mathbf{FSets}_{G_1}, \beta_1), (\mathbf{FSets}_{G_2}, \beta_2))$$

is bijective.

This, in fact, follows from some elementary characterization of profinite groups.

Lemma

G being a profinite group is. for the tautological basepoints $\beta : \mathbf{FSets} \to \mathbf{FSets}_G$, equivalent to

$$\alpha : G \xrightarrow{\sim} \pi_1(\mathbf{FSets}_G, \beta).$$

Grothendieck Conjecture for Connected Anabelioids

As profinite groups are quasi-compact and Hausdorff, any bijection is a homeomorphism. Thus, it suffices to see that it is bijective.

Injectivity is clear: for any open normal subgroup $N \subseteq G$, $\ker(\alpha) \subseteq N$ holds.

Surjectivity is less clear: it suffices to show for any $\gamma \in \pi_1(\mathbf{FSets}_G, \beta)$ and for any open normal subgroup $N \subseteq G$ that the existence of $g \in G$ such that

$$\gamma_{G/N} = \alpha(g)_{G/N}.$$

But this follows by taking g such that $\gamma_{G/N}(N) = gN$ holds.

Grothendieck Conjecture for Connected Anabelioids

In situations where basepoints are not fixed, it is necessary to consider outer morphims.

Definition

1. Define a category $\mathfrak{Hom}^{\mathfrak{Out}}(G, H)$ whose objects are elements of $\mathrm{Hom}(G, H)$, whose morphism $\phi_1 \to \phi_2$ is $h \in H$ such that $\phi_2 = h\phi_1 h^{-1}$.
2. Define a category $\mathfrak{Mor}(\mathbf{C}_1, \mathbf{C}_2)$ whose objects are anabelioid morphisms $\mathbf{C}_1 \to \mathbf{C}_2$ modulo isomorphism.

Theorem

The natural functor

$$\mathfrak{Hom}^{\mathfrak{Out}}(G, H) \to \mathfrak{Mor}(\mathbf{FSets}_G, \mathbf{FSets}_H)$$

gives rise to a category equivalence.

Slimness Implies Rigidity

For a morphism between connected anabelioids $\phi : \mathbf{C}_1 \to \mathbf{C}_2$, define $\mathbf{I}_\phi \subseteq \mathbf{C}_1$ to be the smallest subcategory of $\mathbf{C}_1$ which contains all subquotients of objects in the essential image of ϕ^*. This is a connected anabelioid and once a basepoint of $\mathbf{C}_1$ fixed, for the induced basepoints

$$\pi_1(\mathbf{C}_1) \twoheadrightarrow \pi_1(\mathbf{I}_\phi) \hookrightarrow \pi_1(\mathbf{C}_2)$$

holds.

By using the category equivalence in Grothendieck Conjecture, it is immediate to see:

Corollary (Slimness Implies Rigidity)

Let $\phi : \mathbf{C}_1 \to \mathbf{C}_2$ be a morphism of connected anabelioids, then

$$\mathrm{Aut}(\phi) \simeq Z_{\pi_1(\mathbf{C}_1)}(\pi_1(\mathbf{I}_\phi))$$

holds. Inparticular, morphisms over a slim anabelioid are rigid.

Slimness Implies Rigidity

Let us define a class of anabelioid morphisms which correspond to open immersions $H \hookrightarrow G$.

Definition

A morphism $\mathbf{C}_1 \to \mathbf{C}_2$ is said to be **finite étale** if there exists some object $X \in \mathrm{ob}(\mathbf{C}_2)$ and an equivalence $\mathbf{C}_1 \xrightarrow{\sim} (\mathbf{C}_2)_X$ such that the composite with the morphism $(\mathbf{C}_2)_X \to \mathbf{C}_2$ induced by taking product with X is equal to $\mathbf{C}_1 \to \mathbf{C}_2$.

Slimness Implies Rigidity

The choice of X and $\mathbf{C}_1 \xrightarrow{\sim} (\mathbf{C}_2)_X$ is, in fact, intrinsic to finite étale morphism itself.

Lemma

For a finite étale morphism $\phi : \mathbf{C}_1 \to \mathbf{C}_2$, one can reconstruct $X \in \mathbf{C}_2$ and the equivalence $\mathbf{C}_1 \xrightarrow{\sim} (\mathbf{C}_2)_X$ in a purely categorical fashion.

Proof.

Take the left adjoint functor $\phi_! : \mathbf{C}_1 \to \mathbf{C}_2$ to $\phi^* : \mathbf{C}_2 \to \mathbf{C}_1$. Let $X := \phi_!(T)$ and $\mathbf{C}_1 \to (\mathbf{C}_2)_X$ be the anabelioid morphism obtained by composing the forgetful functor $(\mathbf{C}_2)_X \to \mathbf{C}_2$ and ϕ^*. Here, T is the terminal object in $\mathbf{C}_1$. □

Change of Grothendieck Universe

Let me explain an example of "change of Grothendieck universe".

Definition

Let $\mathbf{C}$ be a slim anabelioid, $\mathfrak{Et}(\mathbf{C})$ be the 2-category whose objects are finite étale covering over $\mathbf{C}$ and morphisms are finite étale morphisms compatible with covering structure morphisms (in the sense of 1-commutativeness).

As $\mathbf{C}$ is slim, finite étale morphisms over it are rigid. Thus, if one considers $\mathbf{Et}(\mathbf{C})$, the coarsification of $\mathfrak{Et}(\mathbf{C})$, i.e., the category whose class of morphisms is the isomorphism class of the original class of morphisms, no information is lost with regards to commutativity of diagrams, i.e., being 1-commutative in $\mathfrak{Et}(\mathbf{C})$ is equivalent to being commutative in $\mathbf{Et}(\mathbf{C})$.

Change of Grothendieck Universe

If $\mathbf{C}$ is assumed to be U-small, $\mathbf{Et}(\mathbf{C})$ is not U-small. However, between these categories, there exists an essentially surjective and fully faithful functor. As $\mathbf{C} \in \mathbf{Et}(\mathbf{C})$, the following theorem asserts that

$$a \in a$$

in some sense.

Theorem

Let $\mathbf{C}$ be a slim anabelioid. Then, the following functor gives rise to a category equivalence:

$$\mathbf{C} \to \mathbf{Et}(\mathbf{C}),\ X \mapsto (\mathbf{C}_X \to \mathbf{C}).$$

Change of Grothendieck Universe

Essential surjectivity is clear by definitions. For fully faithfulness, it is sufficient to see it for connected anabelioilds. First, calculating the category equivalence we saw in Grothendieck Conjecture section, we obtain

$$\mathrm{Hom}_G(H_1, H_2) \simeq$$

$$\{\phi : H_1 \to H_2 : \exists g \in G, H_1 \subseteq gH_2g^{-1}, \phi(-) = g(-)g^{-1}\}$$

and by considering "$\phi \mapsto g \mapsto \overline{g}$" we also obtain (n.b., here one uses the fact $Z_G(H_1) = 1$)

$$\cdots \xrightarrow{\sim} \{\overline{g} \in G/H_2 : H_1 \subseteq gH_2g^{-1}\}$$

but finally this is shown to be isomorphic to

$$\cdots \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{FSets}_G}(G/H_1, G/H_2).$$

The last isomorphism is achieved by considering mapping $\overline{g}$ to the "multiplying by $\overline{g}$-map" .

Further Topics on Galois Categories

One can define fundamental groups for general anabelioids. First, observe that:

Lemma

Let G be a group, $\mathbf{G}_G$ be the corresponding groupoid, then

$$\mathrm{Hom}(\mathbf{G}_G, \mathbf{FSets}) \xrightarrow{\sim} \mathbf{FSets}_G,\ \mathcal{F} \mapsto \mathcal{F}(*)$$

holds.

Now, for an anabelioid $\mathbf{C}$, define its groupoid $\Pi_1(\mathbf{C})$ to be the category whose class of objects is $\mathrm{Hom}_{\mathfrak{Anab}}(\mathbf{FSets}, \mathbf{C})$, and the set of morphisms $\beta_1 \to \beta_2$ is $\mathrm{Isom}_{\mathfrak{Anab}}(\beta_1, \beta_2)$. This turns to be a generalization of the notion of fundamental groups for connected anabelioids to general anabelioid, thet is, for a fixed basepoint β,

$$\mathbf{FSets}_{\Pi_1(\mathbf{C})} := \mathrm{Hom}(\Pi_1(\mathbf{C}), \mathbf{FSets}) \simeq \mathbf{FSets}_{\mathrm{Aut}(\beta)} \simeq \mathbf{FSets}_{\pi_1(\mathbf{C},\beta)}$$

holds.

Further Topics on Galois Categories

Details omitted, but we can consider the dual notion of stack, costack, and the condition that a 2-functor is a costack is a generalization of the property of Seifert-van Kampen-ness.

Theorem ([P16])

Let S be a Noetherian connected scheme. Then,

$$\mathbf{F\acute{e}t}(S) \to \mathbf{Grpd},\ X \mapsto \Pi_1(\mathbf{F\acute{e}t}(X))$$

is a costack.

Further Topics on Galois Categories

Definition

A topological group G is said to be Noohi if for the forgetful functor $\mathcal{F} : \mathbf{Sets}_G \to \mathbf{Sets}$,

$$G \to \mathrm{Aut}(\mathcal{F})$$

is an isomorphism.

By considering Raikov completeness, one clearly sees that profinite groups are Noohi.

Theorem ([BS13])

A category $\mathbf{C}$ is said to be infinite Galois if there exists a Noohi group G and a category equivalence $\mathbf{C} \simeq \mathbf{Sets}_G$. Then, the issue of whether $\mathbf{C}$ is infinite Galois or not is completely determined in a purely category theoretic fashion.

Further Topics on Galois Categories

Finally let me explain the relationship with classical Galois theory. The Galois category over $\mathrm{Spec}(k)$

$$\mathbf{F\acute{e}t}(\mathrm{Spec}(k)) \xrightarrow{\sim} \mathbf{FSets}_{G_k}$$

gives, if restricted to the full subcategory of connected objects,

$$\mathbf{FIMExt}_{k^{\mathsf{sep}}/k} \simeq \mathbf{F\acute{e}t}(\mathrm{Spec}(k))^0 \xrightarrow{\sim} \mathbf{FSets}^0_{G_k} \simeq (\mathbf{SubGr}^{\mathsf{open}}_{G_k})^{\mathsf{op}}.$$

This category equivalence is the Galois correspondence in elementary field theory. That is, classical Galois theory is the theory of Galois equivalence restricted to connected objects.

Categorical Reconstruction of Categories

We also have a categorical reconstruction of categories([StrSpec]). Let $\mathbf{X}$ be a connected category. We shall see that one can reconstruct $\mathbf{X}$ up to op from $\mathbf{Con_X}$.

A constructive functor over a category $\mathbf{X}$ is a functor $\mathcal{F} : \mathbf{F} \to \mathbf{X}$ such that

1. $\mathcal{F}$ is faithful.
2. For every $A \in \mathbf{F}$ and for every isomorphism with domain $\mathcal{F}(A)$, u lifts uniquely to an isomorphism u' with domain A.

Denote the category of constructive functors over $\mathbf{X}$ by $\mathbf{Con_X}$.

A category $\mathbf{C}$ such that there exists a connected category $\mathbf{X}$ and a category equivalence $\mathbf{Con_X} \simeq \mathbf{C}$ is called **of CF type**.

Structure Species

Definition

Let $\mathbf{C}$ be a category. A pair $\Sigma = (E, S)$ is called **structure species** on $\mathbf{C}$ if

1. $E : \mathbf{C} \to \mathbf{Ord}$ and $S : \mathbf{C}^{\simeq} \to \mathbf{Sets}$ are functors.
2. Functorially $\mathcal{D} \circ S \overset{\text{subcategory}}{\subseteq} E \circ \mathcal{J}$.

where $\mathcal{D} : \mathbf{Sets} \to \mathbf{Ord}$ and $\mathcal{J} : \mathbf{C}^{\simeq} \to \mathbf{C}$. A morphism between structure species $\Sigma = (E, S) \to \Sigma' = (E', S')$ is defined to be a morphism of functors $\phi : S \to S'$ such that

$$\forall u \in \mathrm{Hom}_{\mathbf{C}^{\simeq}}(a, b), \forall U \in \mathrm{ob}((\mathcal{D} \circ S)(a)), \forall V \in \mathrm{ob}((\mathcal{D} \circ S)(b)),$$

$$[(E \circ \mathcal{J})(u)(U) < V \implies (E' \circ \mathcal{J})(u)(\mathcal{D}(\phi_a)(U)) < \mathcal{D}(\phi_b)(V)].$$

Denote the category of structure species over $\mathbf{C}$ by $\mathbf{Sp}_{\mathbf{C}}$.

Theorem ([Ab])

$\mathbf{Con}_{\mathbf{C}} \simeq \mathbf{Sp}_{\mathbf{C}}$.

Reconstruction of Objects and Morphisms

Fix an of CF type category $\mathbf{C}$. Then to reconstruct "$\mathbf{X}$" up to opposition it suffices to reconstruct "$\mathbf{X}^{\text{skelton}}$" up to op.

Definition

For a category $\mathbf{C}$ and objects $A, B \in \mathrm{ob}(\mathbf{C})$ define the category

$$\mathbf{Path}_{\mathbf{C}}^{A,B}$$

whose objects are pairs (F, ϕ) such that $\phi : A \coprod B \overset{\text{non-isom}}{\hookrightarrow} F$.

Reconstruction of Objects

First we shall reconstruct “$\mathrm{ob}(\mathbf{X}^{\text{skelton}})$” , “$\mathrm{End}_{\mathbf{X}^{\text{skelton}}}(A)$” , “$\mathrm{Hom}_{\mathbf{X}^{\text{skelton}}}\{A, B\}$” . For an of CF type category $\mathbf{C}$,

1. Define $\mathrm{ob}^{\dagger}_{\mathbf{C}}$ to be the set of minimal objects of $\mathbf{C}$ modulo isomorphism.
2. For $A \in \mathrm{ob}^{\dagger}_{\mathbf{C}}$, let $\mathrm{end}_{\mathbf{C}}(A)^{\dagger} := \mathrm{Aut}_{\mathbf{C}}(A)$.
3. For $A, B \in \mathrm{ob}^{\dagger}_{\mathbf{C}},\ A \neq B$, let $\mathrm{hom}^{\text{non-isom}}_{\mathbf{C}}\{A, B\}^{\dagger}$ be the set of minimal objects (F, ϕ) of $\mathbf{Path}^{A,B}_{\mathbf{C}}$ such that there is no pair

$$(A \to C, B \to D, C \coprod D \to F)$$

   such that $A \coprod B \to C \coprod D \to F$ is equal to ϕ.
4. For $A, B \in \mathrm{ob}^{\dagger}_{\mathbf{C}}$, let

$$\mathrm{hom}_{\mathbf{C}}\{A, B\}^{\dagger} := \begin{cases} \mathrm{end}_{\mathbf{C}}(A)^{\dagger} \coprod \mathrm{hom}^{\text{non-isom}}_{\mathbf{C}}\{A, B\}^{\dagger} & (A = B) \\ \mathrm{hom}^{\text{non-isom}}_{\mathbf{C}}\{A, B\}^{\dagger} & (A \neq B) \end{cases}$$

Reconstruction of Objects

Now we can reconstruct objects and morphisms:

Lemma

For a fixed reference equivalence $\alpha : \mathbf{C} \xrightarrow{\sim} \mathbf{Con}_{\mathbf{X}}$, we may construct canonical bijections

1. $\alpha_{\mathsf{ob}} : \mathrm{ob}_{\mathbf{C}}^{\dagger} \xrightarrow{\sim} \mathrm{ob}(\mathbf{X}^{\mathsf{skelton}})$.
2. $\alpha_{\mathsf{hom}}^{A,B} : \mathrm{hom}_{\mathbf{C}}\{A, B\}^{\dagger} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{X}^{\mathsf{skelton}}}\{\alpha_{\mathsf{ob}}(A), \alpha_{\mathsf{ob}}(B)\}$.

Hence, all that remains is to reconstruct the composition law.

Reconstruction of Composition Law

For simplicity, let

$$a := \alpha_{\mathsf{ob}}(A),\ b := \alpha_{\mathsf{ob}}(B)$$

$$u := \alpha_{\mathsf{hom}}(f), v_1 := \alpha_{\mathsf{hom}}(g_1), v_2 := \alpha_{\mathsf{hom}}(g_2).$$

Then it is immediate to see that:

Lemma

For $A \neq B,\ g_1, g_2 \in \hom_{\mathbf{C}}^{\mathsf{non\text{-}isom}}\{A, B\}^{\dagger}$, TFAE:

1. Either $v_2 = v_1 \circ u$ or $v_2 = u \circ v_1$ holds.
2. Either $g_2 = g_1 \circ (\mathrm{id} \coprod f^{-1})$ or $g_2 = g_1 \circ (\mathrm{id} \coprod f)$ holds.

By this lemma, now we reconstructed the composition law of non-isomorphisms with automorphisms.

Reconstruction of Composition Law

Also of particular importance is:

Corollary

Fixing one $v_1 \in \mathrm{Hom}_{\mathbf{X}^{\text{skelton}}}(a, b)$, for arbitrary

$$g_2 \in \mathrm{hom}_{\mathbf{C}}^{\text{non-isom}}\{A, B\}^{\dagger}$$

we know whether $v_2 \circ v_1$ is definable or not. In particular, as $\mathbf{X}$ is connected, choosing an orientation of a non-isomorphism of the model $\mathbf{X}$ reconstructs whether "which direction is compatible with the morphism chosen", up to op, in a purely category theoretical manner from $\mathbf{C}$.

Reconstruction of Objects

Finally for $g_1 = (F_1, \phi_1) \in \hom^{\text{non-isom}}_{\mathbf{C}}\{A, B\}^\dagger$, $g_2 = (F_2, \phi_2) \in \hom^{\text{non-isom}}_{\mathbf{C}}\{B, C\}^\dagger$, $g_3 = (F_3, \phi_3) \in \hom^{\text{non-isom}}_{\mathbf{C}}\{C, A\}^\dagger$, let us consider their composition laws.

Lemma

TFAE:

1. Either $v_3 = v_2 \circ v_1$ or $v_3 = v_1 \circ v_2$ holds.
2. There exists the colimit F of $F_1 \leftarrow B \rightarrow F_2$ and a morphism $F_3 \rightarrow F$ such that,

$$
\begin{array}{ccccccc}
A & \longrightarrow & A \coprod B & \longrightarrow & F_1 & & \\
& \searrow & & & & \searrow & \\
& & C \coprod A & \longrightarrow & F_3 & \longrightarrow & F \\
& \nearrow & & & & \nearrow & \\
C & \longrightarrow & B \coprod C & \longrightarrow & F_2 & &
\end{array}
$$

commutes.

The End

Thank you for listening!

Bibliography I

M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.

B. Bhatt, P. Scholze, *The pro-étale topology for schemes*, arXiv:1309.1198.

B. Bhatt, P. Scholze, *Prisms and Prismatic Cohomology*, arXiv:1905.08229.

A. A. Blanchard *Structure species and constructive functors*, Canadian J. Math. **26** (1974), 1217-1227.

A. Cadoret, *Galois Categories*, in *Arithmetic and Geometry Around Galois Theory*, Progress in Mathematics, **304**, 171-246, Springer-Verlag, 2013.

P. Deligne, D. Mumford, *The irreducibility of the moduli space of curves of given genus*, IHES Publ. Math., **36** (1969), 75-109.

Bibliography II

- P. Freyd, *Abelian Categories*, Harper, 1966.
- A. Grothendieck, *Revêtements étales et groupe fondamental - S.G.A.1*, Lect. Note in Math., **224**, Springer-Verlag, 1971.
- L. Hesselholt, *Lecture notes on Witt vectores*, available at `http://www.math.nagoya-u.ac.jp/~larsh/papers/s03/wittsurvey.pdf`
- Y. Hoshi, *Introduction to mono-anabelian geometry*, RIMS Preprint **1868**.
- M. Jarden, J. Ritter, *On the characterization of local fields by their absolute Galois groups*, J. Number Theory **11** (1979), no. 1, 1-13.
- P. T. Johnstone, *Topos Theory*, Academic Press, London, 1977.

Bibliography III

K. Joshi, *The absolute Grothendieck conjecture is false for Fargues-Fontaine curves*, arXiv:2008.01228.

M. Kashiwara, P. Schapira, *Categories and Sheaves*, Grundlehren Math. Wiss., 332, Springer, 2006.

S. Mochizuki, *A Version of the Grothendieck Conjecture for p-adic Local Fields*, The International Journal of Math. **8** (1997), 499-506.

S. Mochizuki, *The Geometry of Anabelioids*, Publ. RIMS. Kyoto Univ., **40** (2004), 819-881.

S. Mochizuki, *Categorical Representation of Locally Noetherian Log Schemes*, Adv. Math. **188** (2004), 222-246.

Bibliography IV

- S. Mochizuki, *Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms*, J. Math. Sci. Univ. Tokyo **22** (2015), 939-1156.
- S. Mochizuki, *Inter-universal Teichmüller Theory II: Hodge-Arakelov-theoretic Evaluation*, Publ. Res. Inst. Math. Sci. **57** (2021), 209-401.
- S. Mochizuki, *Inter-universal Teichmüller Theory IV: Log-volume Computations and Set-theoretic Foundations*, Publ. Res. Inst. Math. Sci. **57** (2021), 627-723.
- J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, Grundlehren der Mathematischen Wissenschaften **323** (2000), Springer-Verlag.

Bibliography V

- N. Nikolov, D. Segal, *On finitely generated profinite groups I: strong completeness and uniform bounds*, Ann. of Math. **165** (2007), no. 1, 171-238.
- I. Pirashvili, *The Fundamental Groupoid and the Geometry of Monoids*, PhD Thesis at University of Leicester, 2016; available at `https://lra.le.ac.uk/handle/2381/37837`
- J.-P. Serre, *Abelian ℓ-adic Representations and Elliptic Curves*, W. A. Benjamin, Inc., 1968.
- R. van Dobben de Bruyn, *Automorphisms of categories of schemes*, preprint, arXiv:1906.00921.
- Y. Wakabayashi, *On the category of structure species*, preprint, arXiv:2201.11273.