Def (involution alg)

A : k-alg (not necessarily associative) $\sigma : A \longrightarrow A$: anti-alg map

(A, σ) : involution alg $\overset{\text{def}}{\iff}$ $\sigma^2 = id_A$

Prop (Cayley-Dickson process)

A′の product

(A, σ) : involution alg , $A' := A \times A$, $(a,b)\cdot(c,d) := (ac - d\sigma(b), \sigma(a)d + cb)$

$\sigma' : A' \ni (a,b) \longmapsto (\sigma(a), -b) \in A'$ $\quad 1_{A'} := (1,0)$ とするとき (A', σ') : involution alg

(proof)

A' : k-linear sp は trivial

$$(a,b)\cdot((c,d)+(e,f)) = (a,b)\cdot(c+e, d+f) = (a(c+e) - (d+f)\sigma(b), \sigma(a)(d+f) + (c+e)b)$$

$$\|$$

$$(a,b)\cdot(c,d) + (a,b)\cdot(e,f) = (ac - d\sigma(b), \sigma(a)d + cb) + (ae - f\sigma(b), \sigma(a)f + eb)$$

$$(1,0)\cdot(a,b) = (a,b), \quad (a,b)\cdot(1,0) = (a,b)$$

$$\sigma'((a,b)\cdot(c,d)) = \sigma'((ac - d\sigma(b), \sigma(a)d + cb)) = (\sigma(ac) - \underbrace{\sigma(d\sigma(b))}_{b\sigma(d)}, -\sigma(a)d - cb)$$

$$\|$$

$$\sigma'((c,d))\cdot\sigma'((a,b)) = (\sigma(c), -d)\cdot(\sigma(a), -b) = (\underbrace{\sigma(c)\sigma(a)}_{\sigma(ac)} + b\sigma(-d), \underbrace{\sigma(\sigma(c))}_{c}(-b) - \sigma(a)d$$

$$\sigma'((1,0)) = (\sigma(1), -0) = (1,0)$$

$$\sigma'^2((a,b)) = \sigma'((\sigma(a), -b)) = (\sigma^2(a), -(-b)) = (a,b) \qquad \therefore \sigma'^2 = id_{A'}$$

$\therefore (A', \sigma')$: involution alg ∎

e.g.

$(\mathbb{R}, id_{\mathbb{R}})$ は involution alg である。上記の Cayley-Dickson process を繰り返すことにより、様々な代数系が構成できる。

$(\mathbb{R}\times\mathbb{R}, \sigma)$ を作る。$(a,b), (c,d) \in \mathbb{R}\times\mathbb{R}$ に対して $(a,b)\cdot(c,d) = (ac - db, ad + cb)$

$(a+bi)\cdot(c+di) = (ac - bd) + (ad + bc)i$ より $\mathbb{R}\times\mathbb{R} \ni (a,b) \longmapsto a+bi \in \mathbb{C}$ の同一視により

上記の $\mathbb{R}\times\mathbb{R}$ 上の代数は $\mathbb{C}$ と同型 また $\sigma((a,b)) = (a, -b)$ は $\mathbb{C}$ の共役を取る操作に対応

次に $(\mathbb{C}\times\mathbb{C}, \sigma')$ を作る。$\mathbb{C}\times\mathbb{C}$ の $\mathbb{R}$ 上の基底は $\underbrace{(1,0)}_{1}, \underbrace{(i,0)}_{i}, \underbrace{(0,1)}_{j}, \underbrace{(0,-i)}_{k}$ よりこれらの積が分かれば、$\mathbb{C}\times\mathbb{C}$ 上の積が分かる。

ここで一般に (A, σ) : inv alg に対して A' の積は

$(a, 0)\cdot(b, 0) = (ab, 0)$, $(a, 0)\cdot(0, b) = (0, \sigma(a)b)$ $(0, a)\cdot(b, 0) = (0, ba)$

$(0, a)\cdot(0, b) = (-b\sigma(a), 0)$ であるので

$i\cdot j = (i, 0)\cdot(0, 1) = (0, \sigma(i)) = (0, -i) = k$

$i\cdot k = (i, 0)\cdot(0, -i) = (0, \sigma(i)(-i)) = (0, -1) = -j$

$j\cdot i = (0, 1)\cdot(i, 0) = (0, i)$

$j\cdot k = (0, 1)\cdot(0, -i) = (-(-i)\sigma(1), 0) = (i, 0) = i$

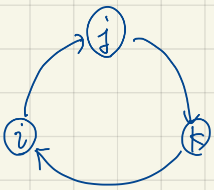
$k\cdot i = (0, -i)\cdot(i, 0) = (0, i(-i)) = (0, 1) = j$

$k\cdot j = (0, -i)\cdot(0, 1) = (-1\sigma(-i), 0) = (-i, 0) = -i$

$i^2 = (i^2, 0) = (-1, 0)$ $j^2 = (0, 1)\cdot(0, 1) = (-1, 0)$

$k^2 = (0, -i)\cdot(0, -i) = (-(-i)i, 0) = (-1, 0)$

	i	j	k
i	−1	k	−j
j	−k	−1	i
k	j	−i	−1



$\mathbb{C}^2 \times \mathbb{C}^2$ をこの積を定めたものを4元数といい $\mathbb{H}$ と書く。$\mathbb{H}$ は積の可換性が成立しない。

また $\mathbb{H}$ 上の involution σ' は

$\sigma'(a+bi+cj+dk) = \sigma'((a+bi, c-di)) = (\sigma(a+bi), -c+di) = (a-bi, -c+di) = a-bi-cj-dk$

次に $(\mathbb{H}\times\mathbb{H}, \sigma'')$ を構成する。$\mathbb{H}\times\mathbb{H}$ の $\mathbb{R}$ 上の基底は

$\underbrace{(1, 0)}_{e_4}$, $\underbrace{(i, 0)}_{e_1}$, $\underbrace{(j, 0)}_{e_5}$, $\underbrace{(k, 0)}_{e_6}$, $\underbrace{(0, 1)}_{e_4}$, $\underbrace{(0, i)}_{e_2}$, $\underbrace{(0, j)}_{e_7}$, $\underbrace{(0, k)}_{e_3}$

$\mathbb{H}\times\mathbb{H}$ に Cayley-Dickson process で積を定めた代数系を $\mathbb{O}$ と書き、8元数という。具体例で計算してみると

$e_6\cdot e_7 = (k, 0)\cdot(0, j) = (0, \sigma(k)j) = (0, -\underbrace{kj}_{-i}) = (0, i) = e_2$

$e_2\cdot e_3 = (0, i)\cdot(0, k) = ((-k)(-i), 0) = (j, 0) = e_5$

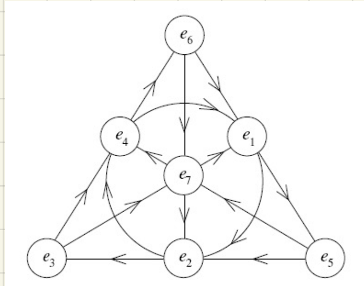
$(e_6\cdot e_7)\cdot e_3 = e_2\cdot e_3 = e_5$

$e_6\cdot(e_7\cdot e_3) = e_6\cdot((0, j)\cdot(0, k)) = e_6\cdot(-k\sigma(j), 0) = (k, 0)\cdot(-i, 0) = (-ki, 0) = (-j, 0) = -e_5$

$\therefore (e_6\cdot e_7)\cdot e_3 \neq e_6\cdot(e_7\cdot e_3)$ となり $\mathbb{O}$ は結合律を満たさないことが分かる。

①の基底の1以外の元の演算表は以下の通りである。

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_4	e_7	$-e_2$	e_6	$-e_5$	$-e_3$
e_2	$-e_4$	-1	e_5	e_1	$-e_3$	e_7	$-e_6$
e_3	$-e_7$	$-e_5$	-1	e_6	e_2	$-e_4$	e_1
e_4	e_2	$-e_1$	$-e_6$	-1	e_7	e_3	$-e_5$
e_5	$-e_6$	e_3	$-e_2$	$-e_7$	-1	e_1	e_4
e_6	e_5	$-e_7$	e_4	$-e_3$	$-e_1$	-1	e_2
e_7	e_3	e_6	$-e_1$	e_5	$-e_4$	$-e_2$	-1



Def

G : group , $\phi : G \times G \times G \longrightarrow k^\times$

ϕ : 3-cocycle $\overset{def}{\Longleftrightarrow}$ $\phi(y,z,w)\phi(x,yz,w)\phi(x,y,z) = \phi(x,y,zw)\phi(xy,z,w)$ $\quad x,y,z,w \in G$

ϕ : normalized 3-cocycle $\overset{def}{\Longleftrightarrow}$ ϕ : 3-cocycle and $\phi(x,e,y) = 1$ $\ {}^\forall x,y \in G$ ← G の unit

$y = e$ のとき $\phi(e,z,w)\phi(x,z,w) = \phi(x,z,w)$ $\therefore \phi(e,z,w) = 1$

$z = e$ のとき $\phi(x,y,w)\phi(x,y,e) = \phi(x,y,w)$ $\therefore \phi(x,y,e) = 1$

e.g.

$G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad \phi(x,y,z) := (-1)^{(x\times y)\cdot z}$ と定めると

ϕ は normalized coboundary 3-cocycle である。

(proof)

${}^\forall x,y,z,w \in G$ をとる。

$\phi(y,z,w)\phi(x,y+z,w)\phi(x,y,z) = (-1)^{(y\times z)\cdot w}(-1)^{(x\times(y+z))\cdot w}(-1)^{(x\times y)\cdot z}$

$= (-1)^{(y\times z)\cdot w + (x\times(y+z))\cdot w + (x\times y)\cdot z}$

$\phi(x,y,z+w)\phi(x+y,z,w) = (-1)^{(x\times y)\cdot(z+w)}\cdot(-1)^{((x+y)\times z)\cdot w} = (-1)^{(x\times y)\cdot(z+w) + ((x+y)\times z)\cdot w}$

ここで, $(x\times y)\cdot z = |x,y,z|$ より $|\ ,\ ,\ |$ は k-linear で交代性を満たすので,

$|x,y,z+w| + |x+y,z,w| = |x,y,z| + |y,z,w| + |x,y,w| + |x,z,w|$

$= |x,y,z| + |y,z,w| + |x,y+z,w|$

また $0 = (0,0,0) \in G$ として ${}^\forall x,y \in G$ に対し, $|x,0,y| = 0$ より

$\phi(x,0,y) = (-1)^0 = 1$ $\quad \therefore \phi$: normalized 3-cocycle

Prop

G: group, ϕ: normalized 3-cocycle on G

$Vect_\phi^G$ を object を G-graded vector sp, morphism を G-graded vector sp の morphism

$V, W \in Vect_\phi^G$ に対して $(V\otimes W)_g = \bigoplus_{\sigma\tau=g} V_\sigma \otimes W_\tau$ として $\otimes$ を定め、

$V, W, Z \in Vect_\phi^G$ に対して $a_{V,W,Z} : (V\otimes W)\otimes Z \ni (v\otimes w)\otimes z \longmapsto \phi(|v|,|w|,|z|)\, v\otimes(w\otimes z) \in V\otimes(W\otimes Z)$

このとき $(Vect_\phi^G, a)$ は monoidal cat

(proof)

$\otimes$ が well-def は やればできる。 $(v\otimes w)\otimes z \in ((V\otimes W)\otimes Z)_g$ のとき、

$\exists\, \sigma, \tau, \kappa \in G$ s.t. $(\sigma\tau)\kappa = g$ かつ $|v| = \sigma$, $|w| = \tau$, $|z| = \kappa$

$\therefore\ a_{V,W,Z}((v\otimes w)\otimes z) \in (V\otimes(W\otimes Z))_g$ $\quad \therefore\ a_{V,W,Z}$ は $Vect_\phi^G$ の morphism

また $f : V_1 \longrightarrow V_2$, $g : W_1 \longrightarrow W_2$, $h : Z_1 \longrightarrow Z_2$ in $Vect_\phi^G$ に対して

$$
\begin{tikzcd}
(V_1\otimes W_1)\otimes Z_1 \arrow[r, "(f\otimes g)\otimes h"] \arrow[d, "a_{V_1,W_1,Z_1}"'] & (V_2\otimes W_2)\otimes Z_2 \arrow[d, "a_{V_2,W_2,Z_2}"] \\
V_1\otimes(W_1\otimes Z_1) \arrow[r, "f\otimes(g\otimes h)"'] & V_2\otimes(W_2\otimes Z_2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(v\otimes w)\otimes z \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & (f(v)\otimes g(w))\otimes h(z) \arrow[d, mapsto] \\
\phi(|v|,|w|,|z|)\, v\otimes(w\otimes z) \arrow[r, mapsto] & \phi(|f(v)|,|g(w)|,|h(z)|)\, f(v)\otimes(g(w)\otimes h(z))
\end{tikzcd}
$$

$\leftarrow\ \phi(|f(v)|, |g(w)|, |h(z)|) = \phi(v, w, z)$ より成立する。

$$
\begin{tikzcd}
((U\otimes V)\otimes W)\otimes Z \arrow[rr, "a_{U\otimes V,W,Z}"] \arrow[d, "a_{U,V,W}\otimes id_Z"'] & & (U\otimes V)\otimes(W\otimes Z) \arrow[d, "a_{U,V,W\otimes Z}"] \\
(U\otimes(V\otimes W))\otimes Z \arrow[r, "a_{U,V\otimes W,Z}"'] & U\otimes((V\otimes W)\otimes Z) \arrow[r, "id_U\otimes a_{V,W,Z}"'] & U\otimes(V\otimes(W\otimes Z))
\end{tikzcd}
$$

$$
\begin{tikzcd}
((u\otimes v)\otimes w)\otimes z \arrow[r, mapsto] \arrow[d, mapsto] & \phi(|uv|,|w|,|z|)\,(u\otimes v)\otimes(w\otimes z) \arrow[d, mapsto] \\
\phi(|u|,|v|,|w|)\,(u\otimes(v\otimes w))\otimes z \arrow[d, mapsto] \arrow[r, phantom, "\circlearrowright"] & \phi(|uv|,|w|,|z|)\,\phi(|u|,|v|,|wz|)\, u\otimes(v\otimes(w\otimes z)) \arrow[d, equal, "\leftarrow \phi:\text{3-cocycle より}"] \\
\phi(|u|,|v|,|w|)\,\phi(|u|,|vw|,|z|)\, u\otimes((v\otimes w)\otimes z) \arrow[r, mapsto] & \phi(|u|,|v|,|w|)\,\phi(|u|,|vw|,|z|)\,\phi(|v|,|w|,|z|)\, u\otimes(v\otimes(w\otimes z)) \in U\otimes(V\otimes(W\otimes Z))
\end{tikzcd}
$$

$$
\begin{tikzcd}
(V\otimes \Bbbk)\otimes W \arrow[rr, "a_{V,\Bbbk,W}"] \arrow[dr] & & V\otimes(\Bbbk\otimes W) \arrow[dl] \\
& V\otimes W &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(v\otimes 1_{\Bbbk})\otimes w \arrow[rr, mapsto] \arrow[dr, mapsto] & \circlearrowright & \phi(|v|,\underbrace{|1_{\Bbbk}|}_{e},|w|)\, v\otimes(1_{\Bbbk}\otimes w) \arrow[dl, mapsto] \\
& v\otimes w &
\end{tikzcd}
$$

∎

<u>Def</u> (abelian 3-cocycle)

G : Abelian grp , ϕ : normalized 3-cocycle on G , $R : G \times G \longrightarrow k^\times$

(ϕ, R) : abelian 3-cocycle $\overset{\text{def}}{\Longleftrightarrow}$ $R(xy,z)\phi(x,z,y) = \phi(x,y,z)R(x,z)\phi(z,x,y)R(y,z)$

$\phi(x,y,z)R(x,yz)\phi(y,z,x) = R(x,y)\phi(y,x,z)R(x,z)$

<u>Def</u> (2-cochain)

G : grp , $F : G \times G \longrightarrow k^\times$: pointwise invertible map

F : 2-cochain on G $\overset{\text{def}}{\Longleftrightarrow}$ $F(e,x) = F(y,e) = 1 \quad (^\forall x, y \in G)$

<u>Prop</u>

G : Abelian group , (ϕ, R) : abelian 3-cocycle on G

$c_{V,W} : V \otimes W \ni v \otimes w \longmapsto R(|v|,|w|)\, w \otimes v \in W \otimes V$

$(\mathrm{Vect}^G_\phi, a, c)$ は braided monoidal cat

(proof)

$f : V_1 \longrightarrow V_2 \quad g : W_1 \longrightarrow W_2$ in Vect^G_ϕ に対して $R(|f(v)|, |g(w)|) = R(|v|,|w|)$ より

$$\begin{tikzcd}
V_1 \otimes W_1 \arrow[r, "f \otimes g"] \arrow[d, "c_{V_1,W_1}"'] & V_2 \otimes W_2 \arrow[d, "c_{V_2,W_2}"] \\
W_1 \otimes V_1 \arrow[r, "g \otimes f"'] & W_2 \otimes V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
v \otimes w \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & f(v) \otimes g(w) \arrow[d, mapsto] \\
R(|v|,|w|)\, w \otimes v \arrow[r, mapsto] & R(|f(v)|,|g(w)|)\, g(w) \otimes f(v) \\
& R(|v|,|w|)\, g(w) \otimes f(v) \arrow[u, equal]
\end{tikzcd}$$

また $|v \otimes w| = |v||w| = |w||v| = |w \otimes v|$
($\uparrow$ G : Abelian より)

また $c^{-1}_{V,W}(w \otimes v) = R(|v|,|w|)^{-1}\, v \otimes w$ とすると

$(c^{-1}_{V,W} \circ c_{V,W}) = id$, $(c_{V,W} \circ c^{-1}_{V,W}) = id$

$$\begin{tikzcd}
W_1 \otimes V_1 \arrow[r, "g \otimes f"] \arrow[d, "c^{-1}_{V_1,W_1}"'] & W_2 \otimes V_2 \arrow[d, "c^{-1}_{V_2,W_2}"] \\
V_1 \otimes W_1 \arrow[r, "f \otimes g"'] & V_2 \otimes W_2
\end{tikzcd}
\qquad
\begin{tikzcd}
w \otimes v \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowright"] & g(w) \otimes f(v) \arrow[d, mapsto] \\
R(|v|,|w|)^{-1}\, v \otimes w \arrow[r, mapsto] & R(|f(v)|,|g(w)|)^{-1} f(v) \otimes g(w) \\
& R(|v|,|w|)^{-1} f(v) \otimes g(w) \arrow[u, equal]
\end{tikzcd}$$

また $R(|v|,|w|) = R(|f(v)|, |g(w)|)$ より

$R(|v|,|w|)^{-1} = R(|f(v)|, |g(w)|)^{-1}$

$\therefore c^{-1}$: natural $\quad \therefore c$: natural iso

$$\begin{tikzcd}
& (W \otimes V) \otimes Z \arrow[r, "a_{W,V,Z}"] & W \otimes (V \otimes Z) \arrow[dr, bend left, "id_Z \otimes c_{V,Z}"] & \\
(V \otimes W) \otimes Z \arrow[ur, "c_{V,W} \otimes id_Z"] \arrow[dr, "a_{V,W,Z}"'] & & & W \otimes (Z \otimes V) \\
& V \otimes (W \otimes Z) \arrow[r, "c_{V,W \otimes Z}"'] & (W \otimes Z) \otimes V \arrow[ur, bend right, "a_{W,Z,V}"'] &
\end{tikzcd}$$

$(v \otimes w) \otimes z \mapsto R(|v|,|w|)\,(w \otimes v) \otimes z \mapsto R(|v|,|w|)\phi(|w|,|v|,|z|)\, w \otimes (v \otimes z) \mapsto R(|v|,|w|)\phi(|w|,|v|,|z|)R(|v|,|z|)\, w \otimes (z \otimes v)$

$(v \otimes w) \otimes z \mapsto \phi(|v|,|w|,|z|)\, v \otimes (w \otimes z) \mapsto \phi(|v|,|w|,|z|)R(|v|,|w||z|)\,(w \otimes z) \otimes v \mapsto \phi(|v|,|w|,|z|)R(|v|,|w||z|)\phi(|w|,|z|,|v|)\, w \otimes (z \otimes v)$

$R(|v|,|w|)\phi(|w|,|v|,|z|)R(|v|,|z|) = \phi(|v|,|w|,|z|)R(|v|,|w||z|)\phi(|w|,|z|,|v|)$ $\circlearrowright$

$$
\begin{tikzcd}[column sep=small]
 & V\otimes(Z\otimes W) \arrow[r, "a^{-1}_{V,Z,W}"] & (V\otimes Z)\otimes W \arrow[dr, bend left, "c_{V,Z}\otimes \mathrm{id}_W"] & \\
V\otimes(W\otimes Z) \arrow[ur, bend left, "\mathrm{id}_V\otimes c_{W,Z}"] \arrow[dr, bend right, "a^{-1}_{V,W,Z}"'] & & & (Z\otimes V)\otimes W \\
 & (V\otimes W)\otimes Z \arrow[r, "c_{V\otimes W,Z}"'] & Z\otimes(V\otimes W) \arrow[ur, bend right, "a^{-1}_{Z,V,W}"'] &
\end{tikzcd}
$$

$$
\begin{aligned}
V\otimes(W\otimes Z) \ni v\otimes(w\otimes z) &\mapsto R(|w|,|z|)\, v\otimes(z\otimes w) \longmapsto R(|w|,|z|)\phi(|v|,|z|,|w|)^{-1}(v\otimes z)\otimes w \\
&\mapsto R(|w|,|z|)\phi(|v|,|z|,|w|)^{-1}R(|v|,|z|)(z\otimes v)\otimes w \in (Z\otimes V)\otimes W \\
&= \phi(|z|,|v|,|w|)^{-1}R(|v||w|,|z|)\phi(|v|,|w|,|z|)^{-1}(z\otimes v)\otimes w \\
v\otimes(w\otimes z) &\mapsto \phi(|v|,|w|,|z|)^{-1}(v\otimes w)\otimes z \mapsto R(|v||w|,|z|)\phi(|v|,|w|,|z|)^{-1}\, z\otimes(v\otimes w)
\end{aligned}
$$

$\therefore (\mathrm{Vect}^G_\phi, a, c)$ は braided monoidal cat ▨

Prop

G: grp, F: 2-cochain on G

$$\phi_{F^{-1}}(x,y,z) := F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y) \qquad ({}^\forall x,y,z\in G)$$

とするとき, $\phi_{F^{-1}}$ は normalized 3-cocycle

(proof)

$$\phi_{F^{-1}}(y,z,w)\phi_{F^{-1}}(x,yz,w)\phi_{F^{-1}}(x,y,z)$$

$$= F(z,w)^{-1}F(yz,w)F(y,zw)^{-1}F(y,z)F(yz,w)^{-1}F(xyz,w)F(x,yzw)^{-1}F(x,yz)F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y)$$

$$\phi_{F^{-1}}(x,y,zw)\phi_{F^{-1}}(xy,z,w)$$

$$= F(y,zw)^{-1}F(xy,zw)F(x,yzw)^{-1}F(x,y)F(z,w)^{-1}F(xyz,w)F(xy,zw)^{-1}F(xy,z)$$

$$\therefore \phi_{F^{-1}}(y,z,w)\phi_{F^{-1}}(x,yz,w)\phi_{F^{-1}}(x,y,z) = \phi_{F^{-1}}(x,y,zw)\phi_{F^{-1}}(xy,z,w)$$

$$\phi_{F^{-1}}(x,e,y) = F(e,y)^{-1}F(x,y)F(x,y)^{-1}F(x,e) = 1_k$$ ▨

Prop

G: Abelian grp, F: 2-cochain on G, $\mathcal{R}_{F^{-1}} : G \times G \ni (x,y) \longmapsto F(x,y)F(y,x)^{-1} \in k^*$

このとき $(\phi_{F^{-1}}, \mathcal{R}_{F^{-1}})$ は abelian 3-cocycle

(proof)

$\phi_{F^{-1}}$ は normalized 3-cocycle であることは既に示した。

$$\mathcal{R}_{F^{-1}}(xy,z)\phi_{F^{-1}}(x,z,y) = F(xy,z)F(z,xy)^{-1}F(z,y)^{-1}F(xz,y)F(x,zy)^{-1}F(x,z)$$

$$\|$$

$$\phi_{F^{-1}}(x,y,z)\mathcal{R}_{F^{-1}}(x,z)\phi_{F^{-1}}(z,x,y)\mathcal{R}_{F^{-1}}(y,z)$$

$$= F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y)F(x,z)F(z,x)^{-1}F(x,y)^{-1}F(zx,y)F(z,xy)^{-1}F(z,x)F(y,z)F(z,y)^{-1}$$

$$\phi_{F^{-1}}(x,y,z)\mathcal{R}_{F^{-1}}(x,yz)\phi_{F^{-1}}(y,z,x)$$

$$= F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y)F(x,yz)F(yz,x)^{-1}F(z,x)^{-1}F(yz,x)F(y,zx)^{-1}F(y,z)$$

$$\|$$

$$\mathcal{R}(x,y)\phi(y,x,z)\mathcal{R}(x,z)$$

$$= F(x,y)F(y,x)^{-1}F(x,z)^{-1}F(yx,z)F(y,xz)^{-1}F(y,x)F(x,z)F(z,x)^{-1} \qquad ▨$$

Def

G: group, F: 2-cochain on G

vector sp $k[G]$ に次の積を定義したものを $k_F[G]$ と書く。

$$x \cdot y = F(x,y)\, xy \qquad (x, y \in G) \quad (xy \text{ は } G \text{ の積})$$

Prop

G: group, F: 2-cochain on $G \Longrightarrow k_F[G]$: algebra in $\mathrm{Vect}^G_{\phi_{F^{-1}}}$

(proof)

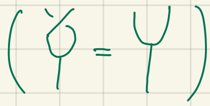
$$\begin{tikzcd}
(k_F[G]\otimes k_F[G])\otimes k_F[G] \arrow[r, "m_{k_F[G]}\otimes id_{k_F[G]}"] \arrow[d, "a_{*,*,*}"'] & k_F[G]\otimes k_F[G] \arrow[dr, bend left] & \\
 & & k_F[G]\otimes k_F[G] \\
k_F[G]\otimes (k_F[G]\otimes k_F[G]) \arrow[r, "id_{k_F[G]}\otimes m_{k_F[G]}"'] & k_F[G]\otimes k_F[G] \arrow[ur, bend right] &
\end{tikzcd}$$

$$\begin{tikzcd}
(x\otimes y)\otimes z \arrow[r, mapsto] \arrow[d, mapsto] & F(x,y)\, xy\otimes z \arrow[r, mapsto] & F(x,y)F(xy,z)\, xyz \arrow[d, equal] \\
F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y)\, x\otimes(y\otimes z) \arrow[r, phantom, "\circlearrowright"] & F(y,z)^{-1}F(xy,z)F(x,yz)^{-1}F(x,y)F(y,z)\, x\otimes yz \arrow[r, mapsto] & F(xy,z)F(x,yz)^{-1}F(x,y)F(x,yz)\, xyz
\end{tikzcd}$$

$$e\cdot x = F(e,x)x = x, \quad x\cdot e = F(x,e)\,x = x \qquad ▨$$

Def (braided commutative)

$(\mathcal{C}, a, c)$: braided monoidal cat, (A, m_A, η_A) : alg in $\mathcal{C}$

A : braided commutative $\overset{\text{def}}{\iff}$ $m_A \circ c_{A,A} = m_A$  ← 絵でかくとこう。

Prop

G : abelian group, F : 2-cochain on G $\Rightarrow$ $k_F[G]$: braided commutative algebra in $\mathrm{Vect}^G_{\phi_{F^{-1}}, R_{F^{-1}}}$

(proof)

$k_F[G]$: alg in $\mathrm{Vect}^G_{\phi_{F^{-1}}}$ は既に示した。

$$(m_{k_F[G]} \circ c_{k_F[G], k_F[G]})(x \otimes y) = m_{k_F[G]}(R_F^{-1}(|x|,|y|)\, y \otimes x)$$

$$= m_{k_F[G]}(F(|x|,|y|)F(|y|,|x|)^{-1}\, y \otimes x) = F(|x|,|y|)F(|y|,|x|)^{-1}F(|y|,|x|)\, yx$$

$$= F(|x|,|y|)\, yx \underset{\uparrow\ G:\text{Abelian より}}{=} F(|x|,|y|)\, xy = m_{k_F[G]}(x \otimes y)$$

$\therefore\ m_{k_F[G]} \circ c_{k_F[G], k_F[G]} = m_{k_F[G]}$ $\quad\therefore\ k_F[G]$: braided commutative alg in $\mathrm{Vect}^G_{\phi_{F^{-1}}, R_{F^{-1}}}$ $\square$

Prop

$F : \mathbb{Z}_2 \times \mathbb{Z}_2 \ni (x, y) \longmapsto (-1)^{xy} \in \mathbb{R}^{\times}$ は 2-cochain on $\mathbb{Z}_2$ で

$\sigma_F : \mathbb{R}_F[\mathbb{Z}_2] \ni x \longmapsto F(x,x)x \in \mathbb{R}_F[\mathbb{Z}_2]$ とすると $(\mathbb{R}_F[\mathbb{Z}_2], \sigma_F)$ は involution alg で、

$\mathbb{R}_F[\mathbb{Z}_2] \cong \mathbb{C}$

(proof)

$F(\bar{0}, x) = (-1)^0 = 1 \quad F(x, \bar{0}) = (-1)^0 = 1$ で F のとりうる値は 1 or -1 より

Fは pointwise invertible より F: 2-cochain on $\mathbb{Z}_2$

$\sigma_F^2(x) = \sigma_F(F(x,x)x) = (-1)^{x^2} F(x,x)x = (-1)^{2x^2} x = 1 \cdot x = x$

$\sigma_F(x \cdot y) = \sigma_F(F(x,y)(x+y)) = F(x,y)F(x+y, x+y)(x+y) = (-1)^{xy+(x+y)^2}(x+y)$

$\qquad\qquad ||$

$\sigma_F(y) \cdot \sigma_F(x) = F(y,y)y \cdot F(x,x)x = F(x,x)F(y,y)F(y,x)(y+x) = (-1)^{x^2+y^2+xy}(x+y)$

$\sigma_F(\bar{0}) = F(\bar{0}, \bar{0})\bar{0} = (-1)^{\bar{0}}\bar{0} = \bar{0}$

$\therefore (\mathbb{R}_F[\mathbb{Z}_2], \sigma_F)$: involution alg

また $^\forall a \cdot \bar{0} + b \cdot \bar{1}, \ c \cdot \bar{0} + d \cdot \bar{1} \in \mathbb{R}_F[\mathbb{Z}_2]$ に対して

$(a \cdot \bar{0} + b \cdot \bar{1}) \cdot (c \cdot \bar{0} + d \cdot \bar{1}) = F(\bar{0}, \bar{0})ac\bar{0} + adF(\bar{0}, \bar{1})\bar{1} + bcF(\bar{1}, \bar{0})\bar{1} + bdF(\bar{1}, \bar{1})\bar{0}$

$= (ac - bd)\bar{0} + (ad + bc)\bar{1}$ となるので $\mathbb{R}_F[\mathbb{Z}_2] \ni a \cdot \bar{0} + b \cdot \bar{1} \longmapsto a + bi \in \mathbb{C}$

また $\sigma_F : \Bbbk_F[\mathbb{Z}_2] \ni a\bar{0} + b\bar{1} \longmapsto aF(\bar{0}, \bar{0})\bar{0} + bF(\bar{1}, \bar{1})\bar{1} = a\bar{0} - b\bar{1} \in \Bbbk_F[\mathbb{Z}_2]$ より

$(\Bbbk_F[\mathbb{Z}_2], \sigma_F) \cong (\mathbb{C}, \sigma)$: as involution alg ▨

Rem

$\mathbb{R}_F[\mathbb{Z}_2]$: braided commutative alg in $\mathrm{Vect}^{\mathbb{Z}_2}_{\phi_{F^{-1}}, a_{F^{-1}}}$ である。

Def (standard involution alg)

G: abelian group, F: 2-cochain on G, $\sigma_F : \Bbbk_F[G] \ni \overset{G\text{の元}}{x} \longmapsto F(x,x)x \in \Bbbk_F[G]$

$(\Bbbk_F[G], \sigma_F)$: standard involution alg $\overset{\text{def}}{\Longleftrightarrow}$ $(\Bbbk_F[G], \sigma_F)$: involution alg

Rem

$(\Bbbk_F[G], \sigma_F)$: standard involution alg $\Longleftrightarrow$ $F(x,y)F(xy,xy) = F(x,x)F(y,y)F(y,x) \quad (^\forall x, y \in G)$

$\qquad F(x,x)^2 = e \quad (^\forall x \in G)$

Prop

G: abelian grp, F: 2-cochain on G s.t. $k_F[G]$: standard involution alg

$\tilde{G} := G \times \mathbb{Z}_2$ とし, $\tilde{F}$: 2-cochain on $\tilde{G}$ を以下で定義すると $k_{\tilde{F}}[\tilde{G}] \cong (k_F[G], \sigma_F)'$ as inv alg

← Cayley-Dickson process

$\tilde{F}((x,\bar{0}),(y,\bar{0})) = F(x,y)$, $\tilde{F}((x,\bar{0}),(y,\bar{1})) = F(x,x)F(x,y)$

$\tilde{F}((x,\bar{1}),(y,\bar{0})) = F(y,x)$, $\tilde{F}((x,\bar{1}),(y,\bar{1})) = -F(x,x)F(y,x)$

(proof)

$k_F[G]'$ の元は $(x,0),(0,y)$ の形の元で生成される $(x,y \in G)$

$\varphi: k_F[G]' \longrightarrow k_{\tilde{F}}[\tilde{G}]$ を $\varphi((x,0)) = (x,\bar{0})$, $\varphi((0,y)) = (y,\bar{1})$ とすると

φ は k-linear iso で,

$k_{\tilde{F}}[\tilde{G}]$ の積は $(x,\bar{0})\cdot(y,\bar{0}) = F(x,y)(xy,\bar{0})$, $(x,\bar{0})\cdot(y,\bar{1}) = F(x,x)F(x,y)(xy,\bar{1})$

$(x,\bar{1})\cdot(y,\bar{0}) = F(y,x)(xy,\bar{1})$, $(x,\bar{1})\cdot(y,\bar{1}) = -F(x,x)F(y,x)(xy,\bar{0})$ より,

$\varphi((x,0)\cdot(y,0)) = \varphi((x\cdot y,0)) = (x\cdot y,\bar{0}) = F(x,y)(xy,\bar{0}) = (x,\bar{0})\cdot(y,\bar{0}) = \varphi((x,0))\cdot\varphi((y,0))$

$\varphi((x,0)\cdot(0,y)) = \varphi((0,\sigma_F(x)\cdot y)) = \varphi((0,F(x,x)x\cdot y)) = F(x,x)(x\cdot y,\bar{1}) = F(x,x)F(x,y)(xy,\bar{1})$

$= (x,\bar{0})\cdot(y,\bar{1}) = \varphi((x,0))\cdot\varphi((0,y))$

$\varphi((0,x)\cdot(y,0)) = \varphi((0,y\cdot x)) = (y\cdot x,\bar{1}) = F(y,x)(\underbrace{yx}_{xy},\bar{1}) = (x,\bar{1})\cdot(y,\bar{0}) = \varphi((0,x))\cdot\varphi((y,0))$

$\varphi((0,x)\cdot(0,y)) = \varphi(-y\cdot\sigma(x),0) = (-y\cdot\sigma(x),\bar{0}) = F(x,x)F(y,x)(-\underbrace{yx}_{xy},\bar{0}) = (x,\bar{1})\cdot(y,\bar{1})$

$= \varphi((0,x))\cdot\varphi((0,y))$

$\varphi((e,0)) = (e,\bar{0})$

また $\varphi(\sigma_F'(x,y)) = \varphi((\sigma_F(x),-y)) = \varphi((F(x,x)x,0)) + \varphi((0,-y)) = F(x,x)(x,\bar{0}) + (-y,\bar{1})$

$\|$

$\sigma_{\tilde{F}}(\varphi(x,y)) = \sigma_{\tilde{F}}((x,\bar{0})) + \sigma_{\tilde{F}}((y,\bar{1})) = \underbrace{\tilde{F}((x,\bar{0}),(x,\bar{0}))}_{F(x,x)}(x,\bar{0}) + \underbrace{\tilde{F}((y,\bar{1}),(y,\bar{1}))}_{\substack{-F(y,y)F(y,y)\\ 1}}(y,\bar{1})$

$\therefore$ φ は involution alg の同型 ▨

e.g.

$G = \mathbb{Z}_2$, $F(x,y) = (-1)^{xy}$ から $\tilde{G} = G \times \mathbb{Z}_2$, $\tilde{F}$: 2-cochain on $\tilde{G}$ を作る操作で

$\tilde{F}((x_1,x_2),(y_1,y_2)) = (-1)^{\tilde{f}((x_1,x_2),(y_1,y_2))}$ とすると

$\tilde{F}((x,\bar{0}),(y,\bar{0})) = F(x,y) = (-1)^{f(x,y)}$ $\quad \therefore \tilde{f}((x,\bar{0}),(y,\bar{0})) = f(x,y)$

$\tilde{F}((x,\bar{0}),(y,\bar{1})) = F(x,x)F(x,y) = (-1)^{f(x,x)+f(x,y)}$ $\quad \therefore \tilde{f}((x,\bar{0}),(y,\bar{1})) = f(x,x) + f(x,y)$

$\tilde{F}((x,\bar{1}),(y,\bar{0})) = F(y,x) = (-1)^{f(y,x)}$ $\quad \therefore \tilde{f}((x,\bar{1}),(y,\bar{0})) = f(y,x)$

$\tilde{F}((x,\bar{1}),(y,\bar{1})) = -F(x,x)F(y,x) = (-1)^{f(x,x)+f(y,x)+1}$ $\quad \therefore \tilde{f}((x,\bar{1}),(y,\bar{1})) = 1 + f(x,x) + f(y,x)$

これらをまとめると $\tilde{f}((x_1,x_2),(y_1,y_2)) = f(x_1,y_1)(\bar{1}-x_2) + f(y_1,x_1)x_2 + f(x_1,x_1)y_2 + x_2y_2$

と書ける。(G,F) から始めて n 回 ~ を繰り返したものを $G^{(n)}$, $F^{(n)}$, $f^{(n)}$ と書くと

$\mathfrak{X} = (x_1,\cdots,x_n)$, $\mathfrak{Y} = (y_1,\cdots,y_n) \in \mathbb{Z}_2^n$ に対して

$$f^{(n)}((\mathfrak{X},x_{n+1}),(\mathfrak{Y},y_{n+1})) = f^{(n-1)}(\mathfrak{X},\mathfrak{Y})(1-x_{n+1}) + f^{(n-1)}(\mathfrak{Y},\mathfrak{X})x_{n+1} + y_{n+1}f^{(n-1)}(\mathfrak{X},\mathfrak{X}) + x_{n+1}y_{n+1}$$

となることが同様の議論により分かる。

$\bar{0}^2 = \bar{0}$ $\quad \bar{1}^2 = \bar{1}$
より $x_1^2 = x_1$

具体的には $f^{(1)}((x_1,x_2),(y_1,y_2)) = x_1y_1(1-x_2) + x_1y_1x_2 + x_1^2y_2 + x_2y_2$

$$= x_1y_1 + (x_1+x_2)y_2$$

$$f^{(2)}((x_1,x_2,x_3),(y_1,y_2,y_3)) = f^{(1)}((x_1,x_2),(y_1,y_2))(1-x_3) + f^{(1)}((y_1,y_2),(x_1,x_2))x_3$$
$$+ y_3 f^{(1)}((x_1,x_2),(x_1,x_2)) + x_3y_3$$

$$= (x_1y_1 + (x_1+x_2)y_2)(1-x_3) + (x_1y_1 + x_2(y_1+y_2))x_3 + y_3(x_1^2 + x_2(x_1+x_2)) + x_3y_3$$

$$= x_1y_1 + x_1y_2 + x_2y_2 - x_1y_1x_3 - x_1y_2x_3 - x_2y_2x_3 + x_1y_1x_3 + x_2y_1x_3 + x_2y_2x_3$$

$-x_1y_2x_3 = x_1y_2x_3 \quad \text{in } \mathbb{Z}^2$

$$+ x_1y_3 + x_1x_2y_3 + x_2y_3 + x_3y_3$$

$$= \sum_{1 \le i \le j \le 3} x_iy_j + y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3$$

ここで $\mathbb{H}$ において $\phi_{F^{(1)}}{}^{-1}(x,y,z) = F^{(1)}(y,z)^{-1}F^{(1)}(xy,z)F^{(1)}(x,yz)^{-1}F^{(1)}(x,y)$ $\qquad x,y,z \in (\mathbb{Z}_2)^2$

$f^{(1)}(y,z) + f^{(1)}(x+y,z) + f^{(1)}(x,y+z) + f^{1}(x,y)$

$= y_1z_1 + (y_1+y_2)z_2 + (\cancel{x_1}+y_1)z_1 + (\cancel{x_1}+y_1+\cancel{x_2}+y_2)z_2 + x_1(\cancel{y_1}+z_1) + (x_1+x_2)(\cancel{y_2}+z_2) + x_1y_1 + (x_1+x_2)y_2 = \bar{0}$

$\therefore \phi_{F^{(1)}}{}^{-1}(x,y,z) = (-1)^{\bar{0}} = 1$ $\quad$ ← Quartanion が associative になる理由

また $R_{F^{(1)-1}}(x, y) = F^{(1)}(x, y) F^{(1)}(y, x)^{-1}$ は

$$f^{(1)}(x, y) + f^{(1)}(y, x) = x_1 y_1 + (x_1 + x_2) y_2 + y_1 x_1 + (y_1 + y_2) x_2 = x_1 y_2 + \underbrace{y_1 x_2}_{-y_1 x_2} = |x, y|$$

ここで ① において、$\phi_{F^{(2)-1}}(x, y, z) = F^{(2)}(y, z)^{-1} F^{(2)}(x+y, z) F^{(2)}(x, y+z)^{-1} F^{(2)}(x, y) \quad (x, y, z \in (\mathbb{Z}_2)^3)$ より

$$f^{(2)}(y, z) + f^{(2)}(x+y, z) + f^{(2)}(x, y+z) + f^{(2)}(x, y)$$

$$= \sum_{1 \le i \le j \le 3} y_i z_j + z_1 y_2 y_3 + y_1 z_2 y_3 + y_1 y_2 z_3 + \sum_{1 \le k \le l \le 3} (x_k + y_k) z_l + \underbrace{z_1 (x_2 + y_2)(x_3 + y_3)}_{z_1 x_2 y_3 + z_1 y_2 x_3} + \overbrace{(x_1 + y_1) z_2 (x_3 + y_3)}^{x_1 z_2 y_3 + y_1 z_2 x_3}$$

$$+ \underbrace{(x_1 + y_1)(x_2 + y_2) z_3}_{x_1 y_2 z_3 + y_1 x_2 z_3}$$

$$+ \sum_{1 \le m \le n \le 3} x_m (y_n + z_n) + (y_1 + z_1) x_2 x_3 + x_1 (y_2 + z_2) x_3 + x_1 x_2 (y_3 + z_3) + \sum_{1 \le s \le t \le 3} x_s y_t + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3$$

$$= x_1 (y_2 z_3 + \underbrace{z_2 y_3}_{-z_2 y_3}) + x_2 (y_3 z_1 + \underbrace{y_1 z_3}_{-y_1 z_3}) + x_3 (y_1 z_2 + \underbrace{y_2 z_1}_{-y_2 z_1}) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = |x \ y \ z|$$

$\mathbb{Z}_2$ の元なので

$$\therefore \phi_{F^{(2)-1}}(x, y, z) = (-1)^{|x, y, z|}$$

また $R_{F^{(2)-1}}(x, y) = F^{(1)}(x, y) F^{(1)}(y, x)^{-1}$ は

$$f^{(2)}(x, y) + f^{(2)}(y, x)$$

$$= \sum_{1 \le s \le t \le 3} x_s y_t + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 + \sum_{1 \le i \le j \le 3} y_i x_j + x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3$$

$$= x_1 y_2 + x_1 y_3 + x_2 y_3 + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 + y_1 x_2 + y_1 x_3 + y_2 x_3$$

$$+ x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3$$

$\therefore \ x = 0$ or $y = 0$ or $x = y$ のとき $R_{F^{(2)-1}}(x, y) = (-1)^0 = 1$

$R_F[\mathbb{Z}_2]$ と $\mathbb{C}$ の対応は $\bar{0} \longleftrightarrow 1, \quad \bar{1} \longleftrightarrow i$ であるので

$\sigma_F(\bar{0}) = \bar{0}, \quad \sigma_F(\bar{1}) = -\bar{1}$ と $\sigma(1) = 1, \quad \sigma(i) = -i$ で $(R_F[\mathbb{Z}_2], \sigma_F) \cong (\mathbb{C}, \sigma)$

$R_{\tilde{F}}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ と $\mathbb{H}$ の対応は $\varphi((x, 0)) = (x, \bar{0}), \quad \varphi((0, y)) = (y, \bar{1})$ より

$\varphi((\bar{0}, 0)) = (\bar{0}, \bar{0}), \quad \varphi((\bar{1}, 0)) = (\bar{1}, \bar{0}), \quad \varphi((0, \bar{0})) = (\bar{0}, \bar{1}), \quad \varphi((0, \bar{1})) = (\bar{1}, \bar{1})$

$1 \longleftrightarrow (1, 0) \longleftrightarrow (\bar{0}, 0) \longleftrightarrow (\bar{0}, \bar{0}) \qquad i \longleftrightarrow (i, 0) \longleftrightarrow (\bar{1}, 0) \longleftrightarrow (\bar{1}, \bar{0})$

$j \longleftrightarrow (0, 1) \longleftrightarrow (0, \bar{0}) \longleftrightarrow (\bar{0}, \bar{1}) \qquad k \longleftrightarrow (0, -i) \longleftrightarrow (0, -\bar{1}) \longleftrightarrow -(\bar{1}, \bar{1})$

また $\mathbb{R}_{\tilde{F}}[\mathbb{Z}_2\times\mathbb{Z}_2]' \longrightarrow \mathbb{R}_{\tilde{F}}[\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2]$ の対応より

$$e_0=(1,0)\longleftrightarrow\big((\bar{0},\bar{0}),(0,0)\big)\longleftrightarrow(\bar{0},\bar{0},\bar{0})$$

$$e_1=(i,0)\longleftrightarrow\big((\bar{1},\bar{0}),(0,0)\big)\longleftrightarrow(\bar{1},\bar{0},\bar{0})$$

$$e_2=(0,i)\longleftrightarrow\big((0,0),(\bar{1},\bar{0})\big)\longleftrightarrow(\bar{1},\bar{0},\bar{1})$$

$$e_3=(0,k)\longleftrightarrow\big((0,0),-(\bar{1},\bar{1})\big)\longleftrightarrow-(\bar{1},\bar{1},\bar{1})$$

$$e_4=(0,1)\longleftrightarrow\big((0,0),(\bar{0},\bar{0})\big)\longleftrightarrow(\bar{0},\bar{0},\bar{1})$$

$$e_5=(j,0)\longleftrightarrow\big((\bar{0},\bar{1}),(0,0)\big)\longleftrightarrow(\bar{0},\bar{1},\bar{0})$$

$$e_6=(k,0)\longleftrightarrow\big(-(\bar{1},\bar{1}),(0,0)\big)\longleftrightarrow-(\bar{1},\bar{1},\bar{0})$$

$$e_7=(0,j)\longleftrightarrow\big((0,0),(\bar{0},\bar{1})\big)\longleftrightarrow(\bar{0},\bar{1},\bar{1})$$

という対応が得られる。

例えば $(e_3\cdot e_4)\cdot e_5 = e_6\cdot e_5 = -e_1$

$$(-1)^{|e_3,e_4,e_5|}\,e_3\cdot(e_4\cdot e_5) = (-1)^{-\begin{vmatrix}\bar{1}&\bar{0}&\bar{0}\\ \bar{1}&\bar{0}&\bar{1}\\ \bar{1}&\bar{1}&\bar{0}\end{vmatrix}}\,e_3\cdot e_7 = (-1)\times e_1$$

$$\therefore\ (e_3\cdot e_4)\cdot e_5 = (-1)^{|e_3,e_4,e_5|}\,e_3\cdot(e_4\cdot e_5)$$

$$(e_3\cdot e_4)\cdot e_3 = e_6\cdot e_3 = e_4$$

$$(-1)^{|e_3,e_4,e_3| \rightsquigarrow 0}\,e_3\cdot(e_4\cdot e_3) = e_3\cdot(-e_6) = e_4$$

$$\therefore\ (e_3\cdot e_4)\cdot e_3 = (-1)^{|e_3,e_4,e_3|}\,e_3\cdot(e_4\cdot e_3)$$

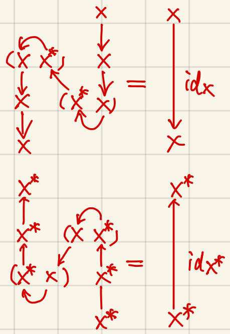
などを確かめることができる。

八元数の quasi-matrix 表現

Def (left dual, right dual)

$(\mathcal{C}, \otimes, a, I, l, r)$: monoidal cat, $X \in Ob(\mathcal{C})$

$X^* \in Ob(\mathcal{C})$: left dual of $X \overset{def}{\Longleftrightarrow} {}^\exists ev_X : X^* \otimes X \longrightarrow I,\ {}^\exists coev_X : I \longrightarrow X \otimes X^*$ s.t.

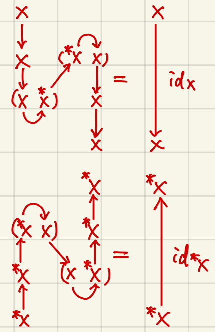


ev$_X$, coev$_X$ は 左の向きに進む

$$\begin{tikzcd} X \arrow[r, "l_X^{-1}"] \arrow[rrrrr, bend right, "id_X"'] & I \otimes X \arrow[r, "coev_X \otimes id_X"] & (X \otimes X^*) \otimes X \arrow[r, "a_{X,X^*,X}"] \arrow[d, phantom, "\circlearrowright"] & X \otimes (X^* \otimes X) \arrow[r, "id_X \otimes ev_X"] & X \otimes I \arrow[r, "r_X"] & X \\ & & \, & & & \end{tikzcd}$$

$$\begin{tikzcd} X^* \arrow[r, "r_{X^*}^{-1}"] \arrow[rrrrr, bend right, "id_{X^*}"'] & X^* \otimes I \arrow[r, "id_{X^*} \otimes coev_X"] & X^* \otimes (X \otimes X^*) \arrow[r, "a^{-1}_{X^*,X,X^*}"] \arrow[d, phantom, "\circlearrowright"] & (X^* \otimes X) \otimes X^* \arrow[r, "ev_X \otimes id_{X^*}"] & I \otimes X^* \arrow[r, "l_{X^*}"] & X^* \\ & & \, & & & \end{tikzcd}$$

${}^*X \in Ob(\mathcal{C})$: right dual of $X \overset{def}{\Longleftrightarrow} {}^\exists ev'_X : X \otimes {}^*X \longrightarrow I,\ {}^\exists coev'_X : I \longrightarrow {}^*X \otimes X$ s.t.



ev$'_X$, coev$'_X$ は 右の向きに進む

$$\begin{tikzcd} X \arrow[r, "r_X^{-1}"] \arrow[rrrrr, bend right, "id_{X^*}"'] & X \otimes I \arrow[r, "id_X \otimes coev'_X"] & X \otimes ({}^*X \otimes X) \arrow[r, "a^{-1}_{X,{}^*X,X}"] \arrow[d, phantom, "\circlearrowright"] & (X \otimes {}^*X) \otimes X \arrow[r, "ev'_X \otimes id_X"] & I \otimes X \arrow[r, "l_X"] & X \\ & & \, & & & \end{tikzcd}$$

$$\begin{tikzcd} {}^*X \arrow[r, "l_{{}^*X}^{-1}"] \arrow[rrrrr, bend right, "id_{{}^*X}"'] & I \otimes {}^*X \arrow[r, "coev'_X \otimes id_X"] & ({}^*X \otimes X) \otimes {}^*X \arrow[r, "a_{{}^*X,X,{}^*X}"] \arrow[d, phantom, "\circlearrowright"] & {}^*X \otimes (X \otimes {}^*X) \arrow[r, "id_{{}^*X} \otimes ev'_X"] & {}^*X \otimes I \arrow[r, "r_{{}^*X}"] & {}^*X \\ & & \, & & & \end{tikzcd}$$

Def (rigid monoidal cat)

$\mathcal{C}$: monoidal cat

$\mathcal{C}$: left (right) rigid $\overset{def}{\Longleftrightarrow}$ ${}^\forall X \in Ob(\mathcal{C})$ に対して left (right) dual が存在する。

$\mathcal{C}$: rigid $\overset{def}{\Longleftrightarrow}$ $\mathcal{C}$: left rigid かつ right rigid

Rem

$\mathcal{C}$: left (right) rigid $\Rightarrow X \in Ob(\mathcal{C})$ に対し、X の left (right) dual は up to isomorphism で unique に定まる。

e.g.

finite dim k-vector sp. のなす圏を $vect_k$ と表記すると、$V \in vect_k$ に対し、

$\{e_i\}$: basis of V, $\{e^i\}$: dual basis of V^* とする。 $ev_V : V^* \otimes V \ni f \otimes v \longmapsto f(v) \in k$
($ev'_V : V \otimes V^* \ni v \otimes f \longmapsto f(v) \in k$)

$coev_V : k \ni 1_k \longmapsto \sum_i e_i \otimes e^i \in V \otimes V^*$ とすると V^* は left dual かつ right dual $\therefore vect_k$ は rigid
($coev'_V : k \ni 1_k \longmapsto \sum_i e^i \otimes e_i \in V^* \otimes V$)

Prop

G : group, ϕ : normalized 3-cocycle $\Rightarrow$ vect^G_ϕ : left rigid monoidal cat

(proof)

vect^G_ϕ は Vect^G_ϕ の subcat より Vect^G_ϕ の monoidal str により、vect^G_ϕ は monoidal cat である。

${}^\forall V \in \mathrm{vect}^G_\phi$ に対し、$\{e_i\}$: basis of V で V_g のいずれかに含まれる basis とする。$|e_i|$ を $|i|$ と表記する。

$\{e^i\}$: dual basis とし、$|e^i| = |i|^{-1}$ とする。$ev_V : V^* \otimes V \ni f \otimes v \longmapsto f(v) \in k$,

$coev_V : k \ni 1_k \longmapsto \sum_i \phi^{-1}(|i|, |i|^{-1}, |i|) e_i \otimes e^i \in V \otimes V^*$ と定めると、ev_V, $coev_V$ は vect^G_ϕ の射である。

$$((id_V \otimes ev_V) \circ a_{V,V^*,V})\Big(\sum_i \phi^{-1}(|i|,|i|^{-1},|i|)(e_i \otimes e^i) \otimes e_j\Big) = \sum_i \phi^{-1}(|i|,|i|^{-1},|i|)\phi(|i|,|i|^{-1},|j|)\delta^i_j e_i = e_j$$

$$\phi(g^{-1},g,g^{-1})\underbrace{\phi(g,g^{-1}g,g^{-1})}_{1}\phi(g,g^{-1},g) = \underbrace{\phi(g,g^{-1},gg^{-1})}_{1}\underbrace{\phi(gg^{-1},g,g^{-1})}_{1} \quad \therefore \phi(g,g^{-1},g)\phi(g^{-1},g,g^{-1}) = 1$$ より

$$((ev_V \otimes id_{V^*}) \circ a^{-1}_{V^*,V,V^*})\Big(e^j \otimes \sum_i \phi^{-1}(|i|,|i|^{-1},|i|)(e_i \otimes e^i)\Big) = \sum_i \phi^{-1}(|i|,|i|^{-1},|i|)\phi^{-1}(|j|^{-1},|i|,|i|^{-1})\delta^j_i e^i$$ ▨

Prop

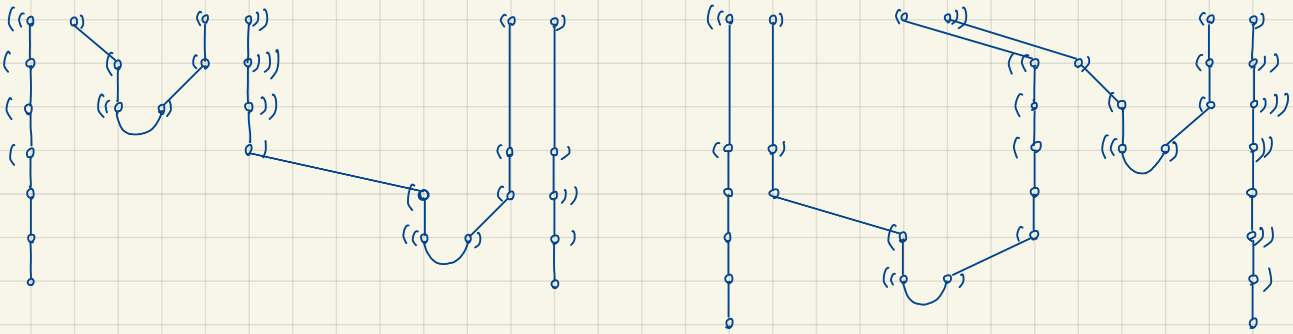
$\mathcal{C}$: left rigid monoidal cat, $V \in Ob(\mathcal{C})$

$$\mu_{V\otimes V^*} : (V \otimes V^*) \otimes (V \otimes V^*) \xrightarrow{a_{V,V^*,V\otimes V^*}} V \otimes (V^* \otimes (V \otimes V^*)) \xrightarrow{id_V \otimes a^{-1}_{V^*,V,V^*}} V \otimes ((V^* \otimes V) \otimes V^*) \xrightarrow{id_V \otimes ev_V \otimes id_{V^*}} V \otimes V^*$$

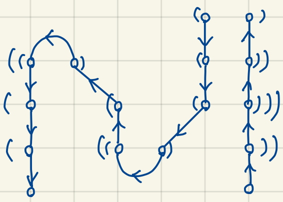
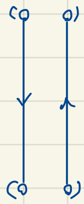
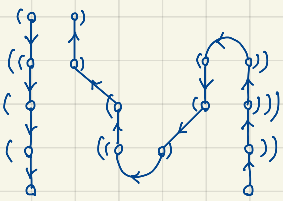
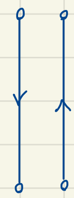
$\eta_{V\otimes V^*} = coev_V : I \longrightarrow V \otimes V^*$ とするとき、$(V \otimes V^*, \mu_{V\otimes V^*}, \eta_{V\otimes V^*})$: alg in $\mathcal{C}$

(proof)

$\mu_{V\otimes V^*} \circ (\mu_{V\otimes V^*} \otimes id_{V\otimes V^*})$ と $\mu_{V\otimes V^*} \circ (id_{V\otimes V^*} \otimes \mu_{V\otimes V^*}) \circ a_{V\otimes V^*, V\otimes V^*, V\otimes V^*}$ を graphical に表現すると、



よって、これらの式は等しい。また、$\mu_{V\otimes V^*} \circ (coev_V \otimes id_{V\otimes V^*})$, $\mu_{V\otimes V^*} \circ (id_{V\otimes V^*} \otimes coev_V)$ を graphical に書くと

 =  $= id_{V\otimes V^*}$  =  $= id_{V\otimes V^*}$ ▨

Prop

$\mathcal{C}$: left rigid monoidal cat , A : alg in $\mathcal{C}$

このとき left A-module $(\triangleright, V)$ と morphism of alg in $\mathcal{C}$ $\rho : A \longrightarrow V\otimes V^*$ は1対1に対応する。

(proof)

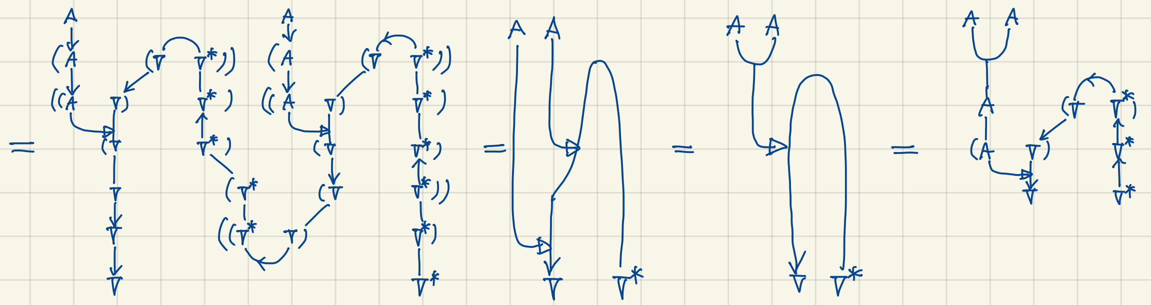
- $(\triangleright, V)$: left A-module とするとき, $\Phi(\triangleright) : A \xrightarrow{id_A\otimes coev_V} A\otimes(V\otimes V^*) \xrightarrow{a^{-1}_{A,V,V^*}} (A\otimes V)\otimes V^* \xrightarrow{\triangleright\otimes id_{V^*}} V\otimes V^*$

と定める。

$$\begin{tikzcd}
A\otimes A \arrow[r, "\Phi(\triangleright)\otimes\Phi(\triangleright)"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "①"] & (V\otimes V^*)\otimes(V\otimes V^*) \arrow[d, "\mu_{V\otimes V^*}"] \\
A \arrow[r, "\Phi(\triangleright)"'] & V\otimes V^*
\end{tikzcd}
\qquad
\begin{tikzcd}
 & k \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_{V\otimes V^*}"] \arrow[d, phantom, "②"] & \\
A \arrow[rr, "\Phi(\triangleright)"'] & {} & V\otimes V^*
\end{tikzcd}$$

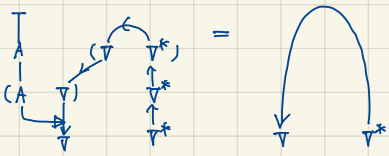
を check する。

$\mu_{V\otimes V^*}\circ(\Phi(\triangleright)\otimes\Phi(\triangleright)) = \mu_{V\otimes V^*}\circ\big(((\triangleright\otimes id_{V^*})\circ a^{-1}_{A,V,V^*}\circ(id_A\otimes coev_V))\otimes((\triangleright\otimes id_{V^*})\circ a^{-1}_{A,V,V^*}\circ(id_A\otimes coev_V))\big)$



$= \Phi(\triangleright)\circ\mu_A$ ···①

$\Phi(\triangleright)\circ\eta_A =$



$= coev_V = \eta_{V\otimes V^*}$ ···②

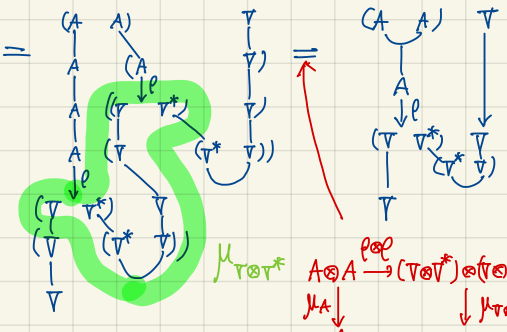
- $\rho : A \longrightarrow V\otimes V^*$: morphism of alg in $\mathcal{C}$ のとき,

$\Phi(\rho) : A\otimes V \xrightarrow{\rho\otimes id_V} (V\otimes V^*)\otimes V \xrightarrow{a_{V,V^*,V}} V\otimes(V^*\otimes V) \xrightarrow{id_V\otimes ev_V} V$ と定める。

$$\begin{tikzcd}
(A\otimes A)\otimes V \arrow[rr, "a_{A,A,V}"] \arrow[d, "\mu_A\otimes id_V"'] & {} \arrow[d, phantom, "①"] & A\otimes(A\otimes V) \arrow[d, "id_A\otimes\Phi(\rho)"] \\
A\otimes V \arrow[r, "\Phi(\rho)"'] & V & A\otimes V \arrow[l, "\Phi(\rho)"]
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes V \arrow[r, "\eta_A\otimes id_V"] \arrow[dr, ""'] \arrow[dr, phantom, shift left=3, "②"] & A\otimes V \arrow[d, "\Phi(\rho)"] \\
 & V
\end{tikzcd}$$

を check する。

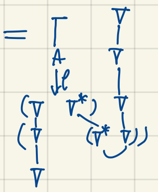
$\Phi(\rho)\circ(id_A\otimes\Phi(\rho))\circ a_{A,A,V} =$



$$\begin{tikzcd}
A\otimes A \arrow[r, "\rho\otimes\rho"] \arrow[d, "\mu_A"'] & (V\otimes V^*)\otimes(V\otimes V^*) \arrow[d, "\mu_{V\otimes V^*}"] \\
A \arrow[r, "\rho"'] & V\otimes V^*
\end{tikzcd}$$

$= \Phi(\rho)\circ(\mu_A\otimes id_V)$ ···①

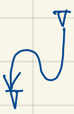
$\Phi(\rho)\circ(\eta_A\otimes id_V) =$



$$\begin{tikzcd}
A \arrow[rr, "\rho"] & {} \arrow[d, phantom, "②"] & V\otimes V^* \\
 & k \arrow[ul, "\eta_A"] \arrow[ur, "\eta_{V\otimes V^*}"'] &
\end{tikzcd}$$

$\eta_{V\otimes V^*} = coev_V$

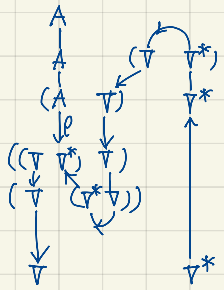
$=$



$= id_V$ ···②

- $\Phi(\Psi(\rho)) = \rho$

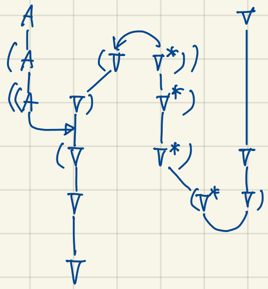
$\Phi(\Psi(\rho)) = \Phi((\mathrm{id}_\nabla\otimes \mathrm{ev}_\nabla)\circ a_{\nabla,\nabla^*,\nabla}\circ(\rho\otimes \mathrm{id}_\nabla)) = (((\mathrm{id}_\nabla\otimes\mathrm{ev}_\nabla)\circ a_{\nabla,\nabla^*,\nabla}\circ(\rho\otimes\mathrm{id}_\nabla))\otimes \mathrm{id}_{\nabla^*})\circ a^{-1}_{A,\nabla,\nabla^*}\circ(\mathrm{id}_A\otimes \mathrm{coev}_\nabla)$

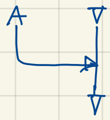
$=$ 

$= \left(A \xrightarrow{\rho} \nabla\otimes\nabla^*\right) = \rho$

- $\Psi(\Phi(\triangleright)) = \triangleright$

$\Psi(\Phi(\triangleright)) = \Psi((\triangleright\otimes\mathrm{id}_{\nabla^*})\circ a^{-1}_{A,\nabla,\nabla^*}\circ(\mathrm{id}_A\otimes\mathrm{coev}_\nabla)) = (\mathrm{id}_\nabla\otimes\mathrm{ev}_\nabla)\circ a_{\nabla,\nabla^*,\nabla}\circ(((\triangleright\otimes\mathrm{id}_{\nabla^*})\circ a^{-1}_{A,\nabla,\nabla^*}\circ(\mathrm{id}_A\otimes\mathrm{coev}_\nabla))\otimes\mathrm{id}_\nabla)$

$=$ 

$=$ 

$= \triangleright$

∎

<u>e.g.</u>

$\mathbb{O}$ は $\mathrm{vect}^{G}_{\phi_{F^{-1}},\alpha_{F^{-1}}}$ の代数より, $\mu_{\mathbb{O}}: \mathbb{O}\times\mathbb{O}\longrightarrow\mathbb{O}$ があるが, $\Phi(\mu_{\mathbb{O}}): \mathbb{O}\longrightarrow \mathbb{O}\otimes\mathbb{O}^*$ は alg map

$\Phi(\mu_\mathbb{O})(e_j) = \left((\mu_\mathbb{O}\otimes \mathrm{id}_{\mathbb{O}^*})\circ a^{-1}_{\mathbb{O},\mathbb{O},\mathbb{O}^*}\circ(\mathrm{id}_\mathbb{O}\otimes\mathrm{coev}_\mathbb{O})\right)(e_j) = \left((\mu_\mathbb{O}\otimes\mathrm{id}_{\mathbb{O}^*})\circ a^{-1}_{\mathbb{O},\mathbb{O},\mathbb{O}^*}\right)\left(\sum_{i=0}^{7} \underbrace{\phi^{-1}(|i|,|i|^{-1},|i|)}_{1}\, e_j\otimes(e_i\otimes e^i)\right)$

$= (\mu_\mathbb{O}\otimes \mathrm{id}_{\mathbb{O}^*})\left(\sum_{i=0}^{7} \underbrace{\phi^{-1}(|j|,|i|,|i|^{-1})}_{1}(e_j\otimes e_i)\otimes e^i\right) = \sum_{i=0}^{7}(e_j\cdot e_i)\otimes e^i$

計算を実行し、行列表示をもとめると,

$$e_0\longleftrightarrow\begin{pmatrix}1&&&&&&&\\&1&&&&&&\\&&1&&&&&\\&&&1&&&&\\&&&&1&&&\\&&&&&1&&\\&&&&&&1&\\&&&&&&&1\end{pmatrix},\ e_1\longleftrightarrow\begin{pmatrix}&-1&&&&&&\\1&&&&&&&\\&&&&-1&&&\\&&&&&&&-1\\&&1&&&&&\\&&&&&&-1&\\&&&&&1&&\\&&&1&&&&\end{pmatrix},\ e_2\longleftrightarrow\begin{pmatrix}&&-1&&&&&\\&&&&1&&&\\1&&&&&&&\\&&&&&-1&&\\&-1&&&&&&\\&&&1&&&&\\&&&&&&&-1\\&&&&&&1&\end{pmatrix},\ e_3\longleftrightarrow\begin{pmatrix}&&&-1&&&&\\&&&&&&&1\\&&&&&1&&\\1&&&&&&&\\&&&&&&-1&\\&&-1&&&&&\\&&&&1&&&\\&-1&&&&&&\end{pmatrix}$$

$$e_4\longleftrightarrow\begin{pmatrix}&&&&-1&&&\\&&-1&&&&&\\&1&&&&&&\\&&&&&&1&\\1&&&&&&&\\&&&&&&&-1\\&&&-1&&&&\\&&&&&1&&\end{pmatrix},\ e_5\longleftrightarrow\begin{pmatrix}&&&&&-1&&\\&&&&&&1&\\&&&-1&&&&\\&&1&&&&&\\&&&&&&&1\\1&&&&&&&\\&-1&&&&&&\\&&&&-1&&&\end{pmatrix},\ e_6\longleftrightarrow\begin{pmatrix}&&&&&&-1&\\&&&&&-1&&\\&&&&&&&1\\&&&&-1&&&\\&&&1&&&&\\&1&&&&&&\\1&&&&&&&\\&&-1&&&&&\end{pmatrix},\ e_7\longleftrightarrow\begin{pmatrix}&&&&&&&-1\\&&&-1&&&&\\&&&&&&-1&\\&1&&&&&&\\&&&&&-1&&\\&&&&1&&&\\&&1&&&&&\\1&&&&&&&\end{pmatrix}$$

行列の積は $\alpha = (\alpha_{ij})$, $\beta = (\beta_{ij})$ に対して, $(\alpha\cdot\beta)_{ij} = \sum_k \phi(|i|,|k|,|k|+|j|)\,\alpha_{ik}\beta_{kj}$ で与えられるので

通常の行列の積とは異なる。

$$e_1\cdot e_2 = \begin{pmatrix}&-1&&&&&&\\1&&&&&&&\\&&&&-1&&&\\&&&&&&&-1\\&&1&&&&&\\&&&&&&-1&\\&&&&&1&&\\&&&1&&&&\end{pmatrix}\begin{pmatrix}&&-1&&&&&\\&&&&1&&&\\1&&&&&&&\\&&&&&-1&&\\&-1&&&&&&\\&&&1&&&&\\&&&&&&&-1\\&&&&&&1&\end{pmatrix} = \begin{pmatrix}&&&&-1&&&\\&&-1&&&&&\\&1&&&&&&\\&&&&&&1&\\1&&&&&&&\\&&&&&&&-1\\&&&-1&&&&\\&&&&&1&&\end{pmatrix} = e_4$$